Controlling the motion of dark solitons by means of periodic potentials: Application to Bose-Einstein condensates in optical lattices

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We demonstrate that the motion of dark solitons (DSs) can be controlled by means of periodic potentials. The mechanism is realized in terms of cigar-shaped Bose-Einstein condensates confined in a harmonic magnetic potential, in the presence of an optical lattice (OL). In the case when the OL period is comparable to the width of the DS, we demonstrate that (a) a moving dark soliton can be captured, switching on the OL, and (b) a stationary DS can be dragged by a moving OL.

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Dark solitons (DSs) are among the most fundamental nonlinear excitations of the nonlinear Schrödinger (NLS) equation with defocusing nonlinearity and have consequently been studied in many fields (see, e.g., [1] for a review). The interest in DSs had been refreshed by the creation of Bose-Einstein condensates (BECs) in ultracold alkali gases [2] and the direct observation of DSs in BECs in a series of experiments [3]. As BECs are confined in harmonic magnetic traps, many theoretical studies dealt with the dynamics of DSs in external potentials [4,5]. In particular, it has been found that in elongated harmonic traps a DS oscillates with frequency $\Omega/\sqrt{2}$, where Ω is the axial trapping frequency. Thermal [6] and dynamical [7] instabilities, mainly referring to rectilinear DSs, have been investigated too. On the other hand, generalizations of the rectilinear DSs, such as ring-shaped DSs, have recently been proposed [8]. Systematic studies of emission of linear waves (sound) by DSs interacting with BEC inhomogeneities, as well as DS-sound interactions, have also been performed [9]. Furthermore, it has been shown that an effectively dissipative effect specific to the BEC, *quantum depletion* of DSs [10], reduces the dark soliton lifetime as atoms tunnel in to fill up the notch at the DS center.

In addition to the above, theoretical and experimental studies of BECs have been performed in the presence of a periodic external potential, in the form of the so-called optical lattice (OL), created by the interference pattern of counterpropagating laser beams [11–13]. The dissipative dynamics of the DSs (including detailed measurements of the sound emitted by the soliton) in a quasi-one-dimensional (Q1D) BEC, confined by a combination of the magnetic trap and the OL, as well as the structure and mobility of DSs in single- and double-periodic OLs, have recently been considered in Refs. [14] and [15], respectively. On the other hand, the stability of DSs in the combined magnetic-OL potential, has been recently studied [16] in the framework of both the discrete and continuum mean-field models.

The objective of this work is to suggest the possibility of controlling the motion of DSs by means of a time-dependent OL periodic potential. In particular, we aim at showing that, in the case when the OL period is comparable to the healing length (that sets the spatial width of the DS), it is possible to (a) snare (immobilize) a *moving* (gray) DS in a local potential well by adiabatically switching the OL on; and (b) capture and drag a stationary (black) DS by a *slowly moving* OL, delivering it to a desired location. To demonstrate these possibilities, we consider the following defocusing NLS [alias Gross-Pitaevskii (GP)] equation in normalized form [2,17,18]:

$$iu_t = -\frac{1}{2}u_{xx} + |u|^2 u + V(x)u, \tag{1}$$

describing the mean-field dynamics of a quasi-one-dimensional cigar-shaped BEC, characterized by its macroscopic wave function $u(x,t)$. The interatomic interactions are considered to be repulsive (positive atomic-scattering length) and the BEC is assumed to be confined by the potential,

$$V(x) = \frac{1}{2}\Omega^2 x^2 + V_0 \cos^2(kx+\theta), \tag{2}$$

where the two terms represent the parabolic magnetic trap and the OL, respectively.

To reduce the original three-dimensional (3D) GP equation to the above 1D form, one requires a very tight radial confinement. In this case, an effective 1D interaction strength, g_{1D}, is obtained upon integrating the 3D interaction strength $g_{3D}=4\pi\hbar^2 a/m$ in the transverse directions (a is the scattering length and m the atomic mass). This yields $g_{1D}=g_{3D}/(2\pi l_\perp^2)$, where $l_\perp=\sqrt{\hbar/m\omega_\perp}$ is the transverse harmonic oscillator length ($\omega_\perp$ is the transverse-confinement frequency). Additionally, to obtain the dimensionless form of Eq. (1), x is scaled in units of the fluid healing length $\xi=\hbar/\sqrt{n_0 g_{1D} m}$ (which also characterizes the size of the dark soliton), t in units of ξ/c (where $c=\sqrt{n_0 g_{1D}/m}$ is the Bogoliubov speed of sound), the atomic density is rescaled by the peak density n_0, and energy is measured in units of the chemical potential of the system $\mu=g_{1D}n_0$. Accordingly, the parameter $\Omega \equiv \hbar\omega_x/g_{1D}n_0$ (ω_x is the confining frequency in the axial direction) determines the magnetic trap strength, while V_0 is the OL strength. Finally, k is the wave number of the OL that can be controlled by varying the angle ψ between the counterpropagating lasers which produce the interference pattern with the wavelength $\lambda \equiv 2\pi/k = (\lambda_{\text{laser}}/2)\sin(\psi/2)$ [19], where λ_{laser} is the wavelength of the laser beams producing the OL and θ is an arbitrary phase constant.

In the absence of external potentials, the defocusing NLS equation (1) gives rise to an exact DS solution [20],

$$u(x,t) = u_0(\cos\varphi \tanh\zeta + i\sin\varphi)\exp(-i\mu_0 t), \tag{3}$$

where u_0 is the amplitude of the uniform background, $\mu_0 \equiv u_0^2$ is the dimensionless chemical potential, φ is the phase shift across the DS ($|\varphi| < \pi/2$), and $\zeta \equiv u_0(\cos\varphi) \times [x - u_0(\sin\varphi)t]$. The amplitude (depth) and velocity of the DS are $u_0\cos\varphi$ and $u_0\sin\varphi$, respectively. The limit case $\varphi = 0$ corresponds to a stationary *black* soliton, $u = u_0\tanh(u_0 x)\exp(-i\mu_0 t)$. In the presence of the potential (2), the background density supporting the DS is nonuniform, and it can be well approximated by the Thomas-Fermi (TF) expression [2],

$$u_{TF} = \sqrt{\max\{0, \mu_0 - V(x)\}}. \tag{4}$$

Here, the chemical potential can be derived by the normalization condition as $\mu_0 \equiv [(3/4\sqrt{2})\Omega Q]^{2/3}$, where $Q = \int_{-\infty}^{-\infty}|u|^2 dx$ is the normalized number of atoms in Eq. (1).

To estimate actual physical quantities, we adopt typical values of the parameters in experiments with DSs in BECs [3]. In particular, for fixed values of the trap strength and normalized chemical potential, $\Omega = 0.025$ and $\mu_0 = 1$, respectively, we assume a cigar-shaped trap with frequencies $\omega_x = 2\pi \times 10$ Hz and $\omega_\perp = 140\omega_x$. Then, for a ^{87}Rb (^{23}Na) condensate, the space and time units are 0.3 μm (2.2 μm) and 0.27 ms (0.56 ms), respectively, the 1D peak density is $5 \times 10^7 \text{m}^{-1} (10^8 \text{m}^{-1})$, and the number of atoms $N \approx 1200$ (17 000).

It is necessary to identify now length scales involved in the problem. First, the parabolic trap strength Ω in Eq. (2) sets the corresponding length scale Ω^{-1}, which we assume to be much larger than the DS width $l_{DS} \equiv (\cos\varphi)^{-1}$ (recall that we have set $u_0^2 = \mu_0 \equiv 1$); otherwise (if the parabolic trap is tight, rather than loose) the DS has no room to exist [16]. Note that, unless the DS is very shallow (i.e., $\cos\varphi \ll 1$), which is not the case of interest, l_{DS} is of the same order of magnitude as the *healing length* ξ. In the framework of Eq. (1), with $\mu \equiv 1$ and $\xi = 1/\sqrt{2}$, we need $\Omega \ll 1$. On the other hand, as concerns the OL potential, both its strength V_0 and wavelength λ may be, generally speaking, arbitrary, hence the ratio of the DS width to λ may take different values. In the following, we will focus on the most interesting case, when the OL period is on the order of the healing length, $\lambda \sim \xi$.

In this case, we expect that the soliton may be trapped at a local extremum of the potential (2), allowing manipulations of the soliton by a time-dependent potential (for instance, dragging the DS by a slowly moving potential). It is necessary to note that the trapped DS is, generally speaking, subject to instabilities [16], which, however, may be weak, hence the instability develops slowly. The potential instabilities are illustrated in Fig. 1, where we depict three different soliton trajectories pertaining to the cases where the soliton is initially placed at or near the bottom of the trap (the parameters are $\Omega = 0.025, V_0 = 0.2$, and $k = 1$). A soliton initially placed exactly at the bottom of the trap with $\theta = \pi/2$ remains trapped for very long times (larger than $t = 1000$, which is the maximal time of the numerical experiment of Fig. 1). A DS, initially placed at a position $x_0(0)$ close to the bottom, escapes only after a fairly long waiting time: $t \approx 500$ for $x_0(0) = 0.01$ (thin solid line) and $t \approx 220$ for $x_0(0) = 0.1$ (solid line). On the other hand, if the phase of the OL is chosen so that $x = 0$ is a local maximum ($\theta = 0$), then a DS placed at $x_0(0) = 0$ escapes relatively quickly ($t \approx 120$, see dashed line). This is consistent with a stability analysis of the stationary solitons for different θ for a wide range of parameter values. In particular, for the parameter values of Fig. 1, a numerical linear stability analysis indicates that if $\theta = 0$, there is a real eigenvalue pair, $\gamma \approx 0.17$, while if $\theta = \pi/2$, there is a much weaker oscillatory instability with $\text{Re}(\gamma) \approx 0.017$ (i.e., with a growth rate ten times smaller than the previous case). Note that, when the initially black DS eventually escapes, it becomes grayer, i.e., its depth decreases and its width increases, due to continuous emission of radiation (sound waves). Thus, the DS gains kinetic energy and starts to perform large-amplitude oscillations.

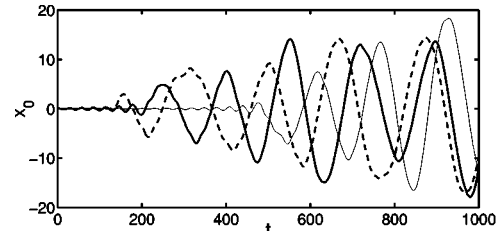

FIG. 1. Motion of the center of a dark soliton in the presence of the magnetic trap with $\Omega = 0.025$ and the optical lattice with $V_0 = 0.2$ and $k = 1$. The three soliton trajectories correspond to different initial positions and optical lattice phases: $x_0(0) = 0.1$ and $\theta = \pi/2$ (solid line), $x_0(0) = 0$ and $\theta = 0$ (dashed line), and $x_0(0) = 0.01$ and $\theta = \pi/2$ (thin solid line).

The above results tend to suggest that it is possible to achieve a "quasitrapping" of the DS in a potential well of the OL, which is a starting point for attempting to control its motion by means of a slowly varying (for instance, moving) OL. Simultaneously, the OL potential will stabilize the motion of the DS against additional small perturbations, such as interaction with a random wave field, that may give rise to a jitter of an unpinned soliton, similar to the well-known jitter of solitons in optical fibers [1] (quite a similar mechanism of the suppression of the jitter of *bright* fiber-optic solitons by an effective periodic potential, which is actually created by a copropagating wave at a different polarization or carrier wavelength, was proposed in Ref. [21]).

We aim to put forward and analyze two concrete possibilities for the manipulation (driving) of DSs by means of nonstationary OLs: (a) snaring and stopping an initially moving DS, which performs large-amplitude oscillations in the parabolic trap, upon switching on an OL, with the objective

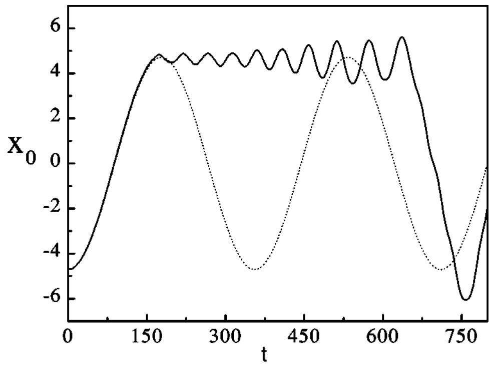

FIG. 2. Entrapment of the moving dark soliton by a potential well belonging to the time-modulated optical lattice. The soliton is initially located at $x_0(0)=-3\pi/2$. The parameters are $\Omega=0.025, V_0=0.07$, $k=1, t_0=177$, and $T=5$. The dotted/solid line corresponds to the DS trajectory in the absence/presence of the time-modulated optical lattice. After getting trapped in the potential, the dark soliton stays there, performing small oscillations, during the time interval $177<t<650$; later, it escapes due to radiation loss.

to trap the DS in a specific well; (b) setting an initially still soliton, trapped in a local well of the OL, in motion, and delivering it at another prescribed location, by a moving OL. Both possibilities are experimentally feasible, as the OL itself can easily be manipulated: It is possible to control its amplitude (V_0) through the intensities of the laser beams that form the lattice [which is relevant to (a)], and it is also possible to induce motion of the OL through time modulation of the phase difference δ between the two counterpropagating beams [this is relevant to (b)], the velocity of the motion being $c=(\pi/k)d\delta/dt$ [19]. Additionally, as we show below, as long as the OL characteristics are varied adiabatically (slowly) in time, the above-mentioned weak instability of the DS trapped near the bottom of the potential well does not disrupt the DS-manipulation mode.

Proceeding to trap the moving DS in a well of the OL, which is adiabatically switched on, we assume an initially black soliton [$\cos\phi=1$ in Eq. (3)] placed away from the center of the trap. In the absence of the OL, the DS oscillates with frequency $\Omega/\sqrt{2}$. The OL is switched on using the following time-dependent potential profile:

$$V_{\rm OL}(x,t) = f(t)V_0\cos^2(kx), \tag{5}$$

where the switching function is chosen as

$$f(t) = \frac{1}{2}\left[1 + \tanh\left(\frac{t-t_0}{T}\right)\right]. \tag{6}$$

The constants t_0 and T in this expression control, respectively, the switch-on time and the width of the time interval over which it occurs.

Simulations of the GP equation with this time-dependent OL potential indeed demonstrate the possibility to snare the initially moving DS in a particular well of the OL, and, subsequently, to hold it there for a relatively long time. A typical example is shown in Fig. 2, where a DS, whose center was initially placed at $x_0(0)=-3\pi/2$, is trapped at the point $x=3\pi/2$, where a local minimum of one of the OL wells is located. The parameters defined in Eq. (6) are $t_0=177$ and $T=5$, the strength of the parabolic trap is $\Omega=0.025$, while the OL parameters are $V_0=0.07$ and $k=1$. As it can be seen in Fig. 2, the soliton remains trapped for a long time, $177<t<650$, and eventually escapes due to radiation losses. It is worth noting that, with the choice of the above parameters, the OL is actually switched on close to the moment in time when the soliton is at the turning point of its trajectory (therefore, it is almost black at the trapping position). Additional simulations have shown that the DS can be snared at any time and at any position (not necessarily close to the turning point), as long as the switch-on time satisfies the condition $T\gtrsim 1$, which is necessary to ensure adiabaticity (otherwise, strong fluctuations of the condensate density are observed).

Now, we proceed to the consideration of the second possibility mentioned above, viz., the transfer of a stationary DS (initially trapped in a well of the OL) by a steadily moving OL. Note that it has been recently shown that an attractive, steadily moving, localized impurity may drag a quasistationary DS [5], but here we aim to consider a more general problem, namely the *targeted transfer* of the DS, via a moving OL.

Following the concept of the robust targeted transfer of solitons in continua or lattices [22], we consider the time-dependent OL potential of the following form:

$$V_{\rm OL}(x,t) = V_0\cos^2[k(x-y(t)) + \pi/2], \tag{7}$$

where the time-varying position $y(t)$, which plays the role of the driver, is chosen as

$$y(t) = \eta_i + \frac{1}{2}(\eta_f - \eta_i)\left[1 + \tanh\left(\frac{t-t_0}{T}\right)\right]. \tag{8}$$

In Eq. (8), η_i and η_f are the initial and (desired) final positions of the DS's center, while T and t_0 are constants controlling, respectively, the duration and the beginning of the transfer [cf. Eq. (6)]. The largest value of the transfer velocity, $v_{\max}\equiv\max(dy/dt)=|\eta_i-\eta_f|/(2T)$, must be sufficiently small to ensure adiabaticity.

Simulations of the GP equation (1) with the OL potential taken as per Eq. (7) demonstrate the possibility of the controlled DS transfer. As shown in Fig. 3, the soliton, initially placed at $\eta_i=-3\pi/2$, is safely delivered to the new location, $\eta_f=\pi/2$ (the parameters are $\Omega=0.025, V_0=0.07, k=1$, $t_0=100$, and $T=60$). We stress that, although the DS oscillates in the OL well, where it was initially captured, it is held in the trapped state, and is dragged by the moving OL quite robustly. When the OL ceases to move, the DS performs small-amplitude oscillations in the well located at the final destination, $x=\eta_f$. The soliton remains well trapped there for a considerable time $(200<t<400)$. However, similar to what was shown in detail above, the radiation loss eventually leads to escape of the DS (for $t>400$). The example shown in Fig. 3 represents a typical targeted transfer. Direct simulations have shown that the proposed process is quite robust for $v_{\max}<0.08$.

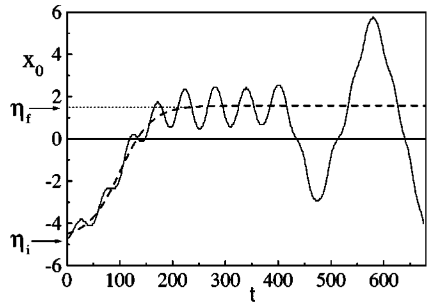

FIG. 3. The targeted transfer of a dark soliton from its initial position, $\eta_i = -3\pi/2$, to the final one, $\eta_f = \pi/2$. The continuous and dashed curves show, respectively, the actual motion law of the trapped dark soliton, and that of the optical lattice drive. The strength of the parabolic trap is $\Omega = 0.025$, the optical lattice parameters are $V_0 = 0.07$ and $k = 1$, and the drive is taken in the form of Eqs. (7) and (8) with $t_0 = 100$ and $T = 60$. The dark soliton delivered by the moving optical lattice to the required destination stays there, performing small oscillations, during the time interval $200 < t < 400$; later, it escapes due to radiation loss.

In conclusion, we have investigated the possibility to control the motion of dark solitons (DSs) by means of periodic optical lattice (OL) potentials. The analysis was based on the one-dimensional nonlinear Schrödinger equation incorporating the parabolic and OL potentials, which is relevant to the case of a cigar-shaped repulsive BEC. We have shown that, in the most interesting case when the OL period is comparable to the DS width, it is possible to (a) trap and nearly stop a moving (gray) DS in a chosen local well of the OL which is adiabatically switched on, and (b) drive a stationary (black) DS by a slowly moving OL, bringing it to a prespecified destination. The results presented are generic (they have been confirmed by many simulations with other realistic parameter values) and predict quite feasible experiments. A limitation on the physical applicability of the results is, however, imposed by dissipative mechanisms in BECs, such as thermal instabilities and quantum depletion (in fact, they limit the DS lifetime). Work is in progress to demonstrate how a similar technique can be used to control the motion of nonlinear excitations, with topological charges, in higher-dimensional atomic condensates.

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