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Regular and chaotic behaviour in an extensible pendulum

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Abstract. The extensible pendulum is studied numerically to illustrate the Hamiltonian transition to chaos. This is an apparently simple system which is well suited to explain concepts related with the onset of chaos. Using Poincaré sections we exhibit the low-energy regular motion and the coexistence of stochastic and regular motion at intermediate energies. We employ other diagnostic techniques for checking our conclusions.

Resumen. Estudiamos numéricamente los movimientos de un péndulo extensible con el propósito de ilustrar la transición hamiltoniana al caos. Este sistema es aparentemente muy simple, pero resulta muy apropiado para la explicación de conceptos relacionados con la aparición del caos. Ilustramos la existencia movimientos regulares a baja energía, asi como la coexistencia de movimientos regulares y caóticos a energías intermedias. Recurrimos a otras técnicas de diagnóstico para corroborar nuestros resultados.

1. Introduction

The properties of the motion of a great number of classical systems can mimic results calculated from a stochastic process. This should be clear from the well known lack of predictability in the tossing of a coin (which is almost the universal paradigm of randomness) or the spinning of a roulette wheel. As prediction is impossible despite the determinism of the basic dynamical rules, this behaviour has been termed *deterministic* chaos. Its development can be considered akin, in a not very precise way of thinking, to the onset of turbulence in a fluid in motion. Over the last few years there have been many contributions which illustrate the transition to chaos in systems with physical interest (Núñez-Yépez *et al* 1989 and references therein, Bercovich *et al* 1991) but most of these have dealt with dissipative chaos, that is, systems where the effects of viscosity, friction, or any other causes of dissipation are important. Stochastic behaviour may also occur as *Hamiltonian chaos* in conservative systems. In fact, the stochastic properties of motion were first discovered in Hamiltonian systems through the prodigious intuition of Poincaré (1892), although its rediscovery for the majority of the scientific community was only 30 years ago in a paper by Hénon and Heiles (1964).

It is the purpose of this paper to discuss and exemplify some basic properties of the Hamiltonian transition to chaos using the apparently simple system of two parametrically coupled oscillators. This system is called the extensible or elastic pendulum (Breitenberger and Mueller 1981, Witt and Gorelik 1933). The system has only two degrees of freedom but displays an extremely rich dynamical behaviour (Broucke and Baxa 1973, Hitzl 1974, Núñez-Yépez *et al* 1990). After giving the motivation for the study of this system from the point of view of molecular vibrations and deriving a few analytical results, we exhibit, with the help of the Poincaré sections, the coexistence of chaotic and regular motion in its dynamics. We next evaluate its Liapunov spectrum. We also use other indicators of the behaviour in order to clarify the meaning of the dynamical complexity found. There is a great deal of aesthetic charm to be derived from this sort of studies which can be shared with students of all levels. One may

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wonder then why this beautiful aspect of dynamics had not been often observed much earlier. An answer to this question appears to be that, although the behaviour was there, no one knew what questions to ask or how to look for it. This was partly due to the enormous faith put in the long term predicting powers of dynamics—summarized in the famous Einstein dictum 'God does not play dice'—and to the lack of computational power since; as we hope to make clear in this work, computers can play a fundamental role in these studies.

2. Hamiltonian systems: integrable and chaotic

A time-independent, or autonomous as it is also called, Hamiltonian system with N degrees of freedom is characterized by its Hamiltonian function

$$H(p,q)=\frac{p^2}{2m}+V(q), \tag{1}$$

where p and q are shorthand for the N canonical coordinates and the N momenta conjugate to them, and $V(q)$ is the potential energy function of the system. The corresponding equations of motion are the $2N$ first-order Hamilton equations

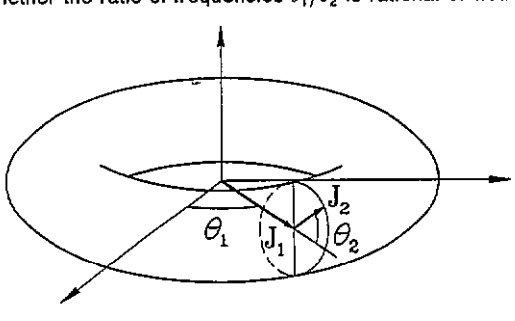
$$\dot{q}_a=\frac{\partial H}{\partial p_a}, \quad \dot{p}_a=-\frac{\partial H}{\partial q_a}, \quad a=1,\ldots,N, \tag{2}$$

which govern the dynamics in the $2N$-dimensional phase space. The solutions to the canonical equations have quite interesting properties. For example, according to Liouville theorem (Landau and Lifshitz 1977), they preserve the phase-space measure or 'volume.' This means that if you take a set of points with measure V_0 as initial conditions, the volume occupied by these points at any later time, as they evolve according to (2), is always the same. If we think of all the trajectories evolving from the original volume as a kind of fluid, this result may be stated saying that the phase fluid is incompressible. This property has deep consequences, for instance it rules out the existence of attractors or repellors of any sort in conservative systems. This is perhaps the most important difference between Hamiltonian and dissipative chaos. Liouville's property also implies the Poincaré-Zermelo recurrence theorem (Arnold 1978), which essentially says that, given a long enough time interval, generic bounded motions in Hamiltonian systems always return 'sufficiently near' to their initial conditions.

A Hamiltonian system is said to be integrable when there are at least N globally defined functions $I_a(p,q)$, $a=1,\ldots,N$, which are independent, single valued, isolating constants of motion in involution, that is, such that their pairwise Poisson brackets vanish $\{I_a,I_b\}=0$, $a,b=1,\ldots,N$ (Martínez-y-Romero *et al* 1992, 1993). If a Hamiltonian system is integrable its associated Hamilton–Jacobi equation is separable in at least one system of coordinates (Landau and Lifshitz 1977). Much of the students intuition on the possible behaviour of a Hamiltonian mechanical system is based on experience with integrable systems, which in many respects behave just like an N-dimensional harmonic oscillator with an energy-dependent frequency. Examples of integrable systems are very rare in classical mechanics; besides the harmonic oscillator and one-dimensional problems, we can only mention the Kepler problem and the Toda lattice (Gutzwiller 1990). For a catalogue of other integrable Hamiltonian systems in three dimensions see Evans (1990). Since in the integrable problems the Hamilton–Jacobi equation is separable, there must exist a canonical transformation which changes the system from the original coordinates to the new action J_i and angle θ_i coordinates. In these coordinates it becomes obvious that an integrable system admits only periodic or conditionally periodic motions. The phase-space trajectories of these systems are then easily seen to be confined to the surface of nested N-dimensional tori which completely fill the available phase space. An obvious result is that there is no stochastic behaviour or chaos in integrable systems; on the tori, the motion can be periodic or conditionally periodic depending on whether the frequencies with which the torus is traversed are rationally dependent or not (see figure 1). The assumption that these two are the only possibilities for the motions in a generic Hamiltonian system, as was indeed implicitly assumed by most people until recently, excludes chaotic motions and is wrong.

What happens when we have an arbitrary Hamiltonian system where the only obvious constant is the Hamiltonian itself? If there are indeed no more constants of motion, the behaviour of the system is not further 'restrained' in any way. There are no globally defined tori tiling the whole phase space; hence, the possible motions may wander over all the available phase space. If the phase space region which these motions fill is $(2N-1)$-dimensional,

Figure 1. If a system is integrable its phase space is foliated by tori. Illustrated here is a two-dimensional torus showing the action-angle variables needed for its specification. The phase-space trajectories of integrable systems are confined to the surface of the torus. The motion is periodic or quasiperiodic depending on whether the ratio of frequencies $\dot{\theta}_1/\dot{\theta}_2$ is rational or not.



the motion appears to have stochastic characteristics. The important point here is that, in the absence of at least N constants of motion, the system begins to show a great sensitivity to the initial conditions; when the system develops this extreme sensitivity, even initially small perturbations (produced, say, by the external environment) are amplified by the dynamics, leading inexorably to catastrophic growth in uncertainty and to the seemingly stochastic properties typical of chaos. Notice that for this to happen it is necessary (but not sufficient) to have a non-linear evolution equation. Most non-linear systems have equations of motion that are not exactly solvable. Hence, to study their behaviour numerical methods are required. As the accuracy of computer numbers is finite, the precision of the initial conditions one can give is also finite. Any initial error is magnified by the dynamical evolution and any long term prediction is impossible even with the help of a computer. Moreover, as in a bounded system there is a finite measure of phase space available to the system, this amplification is by necessity accompanied by a process of contraction giving a very complex geometric structure to the phase-space trajectories.

3. The extensible pendulum

To exemplify the transition to stochastic behaviour we have chosen an apparently simple system with only two degrees of freedom—but then it has the minimum number of degrees of freedom to allow chaotic behaviour in a time-independent Hamiltonian system—the so-called elastic or extensible pendulum. This problem was introduced long ago by Witt and Gorelik (1933) and has been studied by several authors (Contopoulos 1963, Broucke and Baxa 1973, Hitzl 1974, 1975, Breitenberger and Mueller 1981, Núñez-Yépez *et al* 1990, Cuerno *et al* 1992) as a kind of paradigm of non-linear systems. The interest for it comes from various sources, both with mathematical and physical interest (McLachlan 1947, Pippard 1989, Rott *et al* 1991, Aničin *et al* 1993). One of the most interesting suggestions is that it can serve as a classical analogue of a quantum resonance phenomenon observed in some triatomic molecules. If we imagine such a molecule oscillating as in figure 2(a), it is not difficult to convince oneself that a classical model for those motions is a pendulum whose cord has springlike characteristics. The anharmonic coupling so obtained has been of help in explaining the Fermi resonance observed in the infrared spectrum of molecules like CO_2 (Kuzmin and Stuchebrukhov 1989). A classical image of this molecular coupling is then as shown in figure 2(b). The extensible pendulum is thus (Witt and Gorelik 1933, Pippard 1989, Breitenberger and Mueller 1981) time-independent, conservative and reduces to a pair of uncoupled

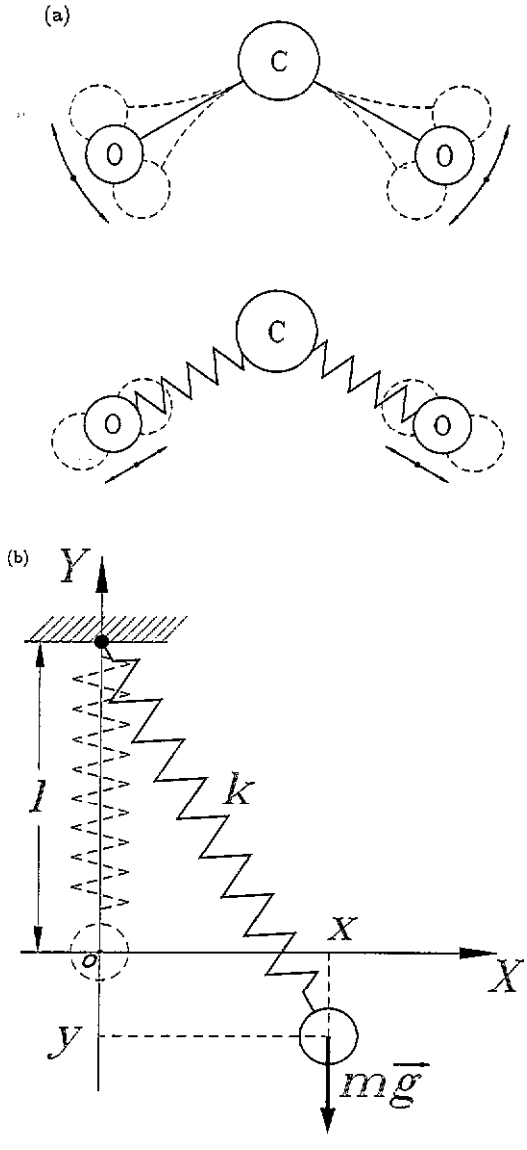


Figure 2. (a) Schematic representation of a CO_2 molecule illustrating the bending and stretching modes of vibration. When the ratio of frequencies is 2, the coupling between these two modes produces a strong parametric resonance known, quantum mechanically, as the Fermi resonance. (b) The extensible pendulum showing the Cartesian coordinates used for its description. The origin is located at the equilibrium position of the system with the mass attached, marked *o* in the figure. Notice that the system can be considered as a model for the coupling of the bending and stretching modes in a molecule like that shown in figure 2(a).

harmonic oscillators for infinitesimal amplitudes. For not so small amplitudes the system seesaws parametrically in amplitude: the stretching mode periodically alters the pendulum length, and hence

the pendulum period also changes with time. This is an inherently non-linear process since the motion cannot be represented as a simple superposition of the two modes of figure 2(a).

The Hamiltonian of the system is found to be (Breitenberger and Mueller 1981, Carretero-González 1992, Aničin *et al* 1993):

$$H(q_1,q_2,p_1,p_2) = \tfrac{1}{2}(p_1^2+p_2^2) + \tfrac{1}{2}((1-c)q_1^2+q_2^2-cq_1^2q_2), \tag{3}$$

where c is a non-dimensional parameter defined as

$$c \equiv 1-\frac{mg}{kl} = 1-\left(\frac{\omega_p}{\omega_s}\right)^2, \tag{4}$$

m is the mass of the pending object, g is the acceleration produced by the gravitational field, l is the length of the spring at equilibrium under a static load mg, ω_s and ω_p are, respectively, the small oscillation frequencies of the spring and pendulum modes, and we have introduced the scaled coordinates $q_1 \equiv x/l$, $q_2 \equiv y/l$, the time has been scaled in units of ω_s^{-1}, and the energy in units of $m\omega_s^2 l^2$. This rescaling manages to reduce the four original parameters of the system (m,g,k,l), to a single one, c. It suffices then to vary c, instead of varying m, g, k and l separately, to obtain all possible dynamical cases. With the mechanical interpretation given for the Hamiltonian (3), we have $\omega_p \le \omega_s$ (Breitenberger and Mueller 1981, Núñez-Yépez *et al* 1990), this implies the restriction $0 \le c \le 1$ on the possible values of c.

The potential energy function appearing in (3), is not confining at all energies. There exists a maximum value of the energy, E_{max}, up to which the particle remains trapped; at greater energies the system 'ionizes.' To evaluate E_{max}, we have to calculate the position of the (only two) saddle points of V, which are $q_{sp}^{\pm} = (\sqrt{2(1-c)}/c,\ \pm(1-c)/c)$, and evaluate the energy there. The symmetry of the potential about the q_1-axis is evident from the equation (3), so the two saddle points have the same energy, which is

$$E_{max} = V(q_{sp}^{\pm}) = \frac{(1-c)^2}{2c^2}. \tag{5}$$

From equation (3) we can see that $E_{min} = 0$, so the energy range of interest to us here is $0 \le E \le (1-c)^2/2c^2$. The low-energy motion of the system has been discussed using the slow-fluctuation technique by Breitenberger and Mueller (1981) and more recently by Aničin *et al* (1993) using perturbation methods, in two interesting papers.

The equations of motion following from (2) and the Hamiltonian (3) are

$$\dot{q}_1 = p_1, \qquad \dot{p}_1 = (c-1)q_1 + cq_1q_2, \tag{6}$$

and

$$\dot{q}_2 = p_2, \qquad \dot{p}_2 = -q_2 + cq_1^2/2. \tag{7}$$

These equations are easily analysed in two cases: $c = 0$ and $c = 1$. The first case reduces the system to two uncoupled oscillators, whereas the second case corresponds to unbounded motion and, therefore, it is of no interest in this work. For other values of the parameter, the equations cannot exactly be solved despite their simple appearance, hence we use a numerical method to integrate the equations of motion. For the numerical computations a 3900 micro Vax computer and a fourth-order Runge–Kutta integration method have been used, but even a personal computer may be used to do many of the computations. Notice that if $c \neq 0$ and $c \neq 1$, there are no obvious global constants of motion in our system excepting the Hamiltonian itself; thus we expect stochastic behaviour in the bounded motions of the extensible pendulum. This initial guess is corroborated by the numerical results shown in the next section. We must emphasize that there is no known method for predicting if a given non-linear Hamiltonian system with two or more degrees of freedom may behave chaotically or not.

4. Regular and chaotic behaviour in the extensible pendulum

4.1. Poincaré sections

One of the most illustrative methods for exhibiting the dynamics of a Hamiltonian system is the so called Poincaré section method. These sections are constructed as the points where a phase-space trajectory pierces, when it goes in a given direction (for example, as we choose here, when $p_2 > 0$), a previously selected plane (P). The plane, called the Poincaré plane, has to be chosen in a way that the trajectory can intersect it several times. In this way, though the trajectories of a conservative N-degrees-of-freedom system 'live' in a $(2N-1)$-dimensional space, we can effectively reduce the dimension to $2N-2$ in order to plot the behaviour of the trajectories. In this way we are able to reduce the phase space trajectory to a discrete mapping, the so-called Poincaré mapping. It is clear that this technique is specially useful in systems like the extensible pendulum, with only two degrees of freedom.

In this paper we are working with a two-dimensional system whose phase space variables are (q_1,q_2,p_1,p_2). As the energy is a constant of motion, the number of variables of our system is reduced from four to three (q_1,q_2,p_1)—p_2 can be calculated from those three using the energy equation. If another independent constant of motion exists, then the trajectory would lay on a two-dimensional surface (i.e. on the intersection of the 3-surface of constant energy and the 3-surface

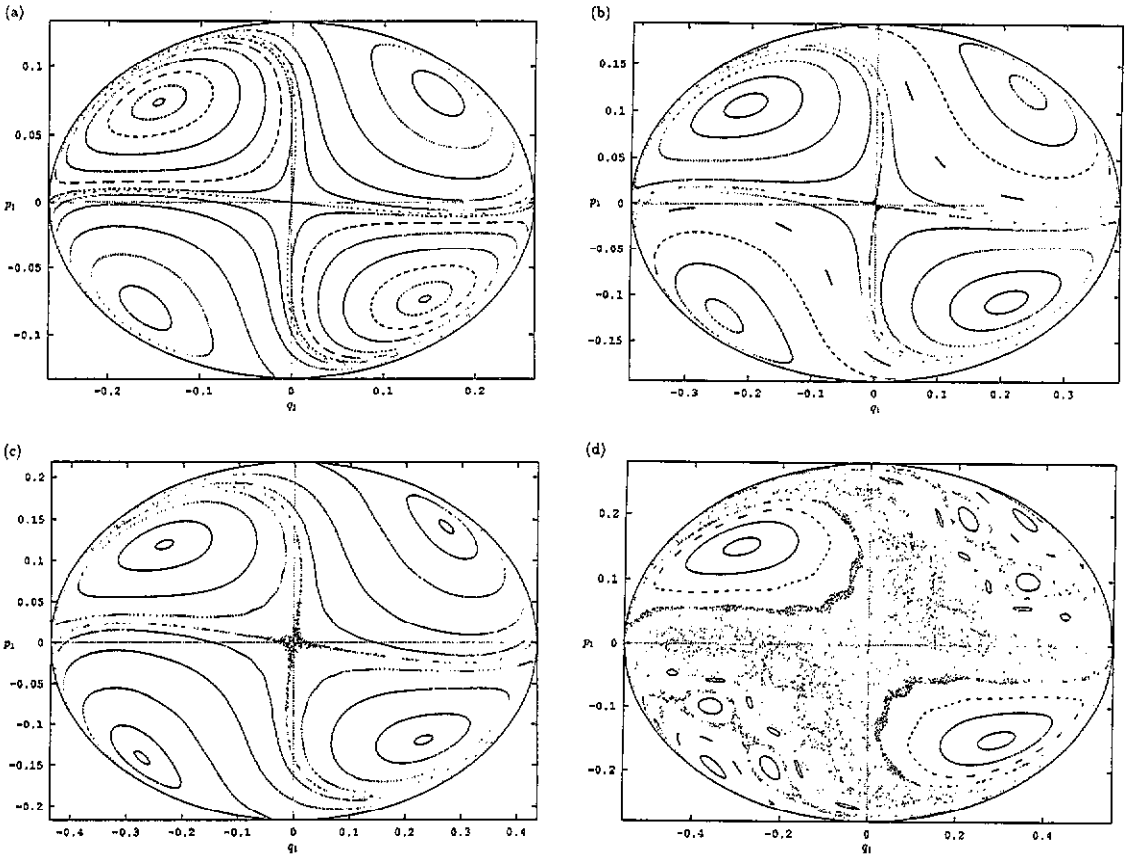
associated with the constancy of the second quantity) and thus the points intersecting the Poincaré section would lay on a smooth curve. This corresponds to the case of a regular trajectory. What happens in the case of an irregular trajectory? In this case no extra constant of motion exists, the trajectory lives then on the three-dimensional hypersurface of constant energy and its intersection with the Poincaré plane would tend to fill a certain two-dimensional region of it. Summarizing, on the Poincaré section the regular orbits appear as points on a smooth curve, whereas any chaotic trajectory appears as a splatter of points filling a certain area of it (see figure 3).

We can observe the transition to chaos in our system in figure 3. This figure shows Poincaré sections calculated, with $c = 0.75$, at four different energies. In this case the Poincaré section is chosen to be the plane (q_1, p_1). The range of energies in which the motion is bounded is $0 \leq E \leq \frac{1}{18}$. The successive Poincaré sections illustrate the onset of stochastic behaviour as the energy is increased. For $E = 0.00875$ (figure 3(a)) only regular orbits (i.e. points located on smooth curves) can be seen; but, at $E = 0.01875$ (figure 3(b)) the central orbit has 'thickened' and a splatter of points that fills a surface on the Poincaré plane can be discerned near the origin; the central region is behaving chaotically. As we continue to increase the energy (figures 3(c) and (d)), the chaotic region grows until we almost reach $E_{\max}$ where most of the Poincaré section is occupied by chaotic orbits. For the purposes of comparison figure 4 illustrates the difference between a regular and a chaotic orbit in the 'physical' space (q_1, q_2).

But how is it that, at low energies, the trajectories seem to be confined to apparently smooth curves if, according to our previous discussion, the only constant of motion is the Hamiltonian? An answer to this question may be given in two related forms:

(1) If the system is not integrable, the only *globally defined, time-independent* and *isolating* constant of motion is H, but there can be other constants which though not globally defined may serve to partially restrain the behaviour of the system (Martínez-y-Romero *et al* 1992, 1993). This is clearly seen in figure 4, where these constants are increasingly less

Figure 3. Poincaré sections for different values of the energy: (a) $E = 0.00875$; (b) $E = 0.01875$; (c) $E = 0.02375$; (d) $E = 0.03875$. The value of the parameter is $c = 0.75$, the same in all cases. The continuous oval-shaped curve corresponds to the boundary of the energetically allowed region on the Poincaré plane.



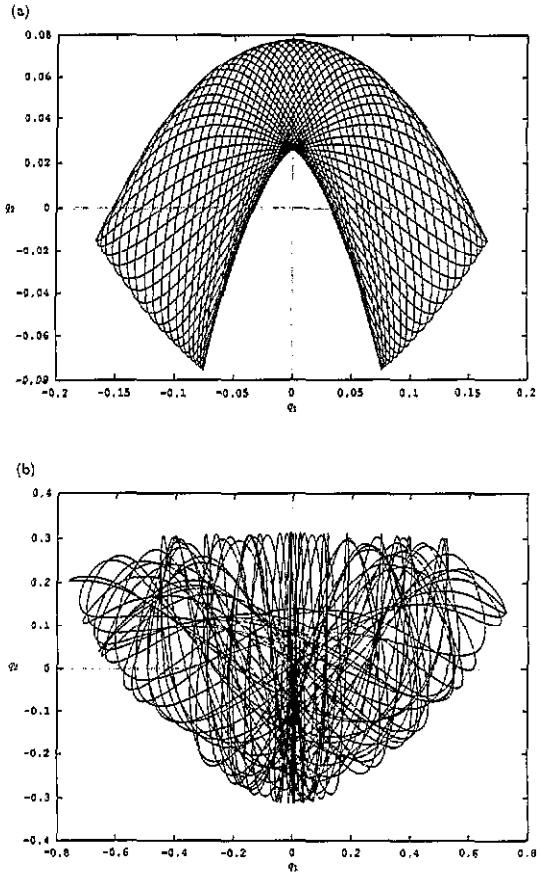


Figure 4. *Trajectories of the system in the physical space* $q_2 - q_1$. (a) Trajectory in a regular region ($E = 0.00375$); (b) trajectory in a chaotic region ($E = 0.04875$).

efficient for smoothing the motion as the energy increases. To convince oneself of the existence of this kind of constants, think on the $2N$ initial conditions, which though well defined, are, in general, not time-independent nor isolating.

(2) The low-energy behaviour of the Poincaré sections illustrates the famous KAM theorem. This theorem essentially states that in slightly perturbed originally integrable systems, not all the tori are destroyed by the perturbation but some are only distorted. What we are exhibiting in figure 3(a) are sections of the surviving tori at very low energies, that is at very small perturbations. The sections also illustrate that even close to E_{max} not all the tori are destroyed. The KAM theorem is not easy to state precisely and there are no simple proofs of it, however good discussion can be found in Berry (1978), Arnold (1978) or Tabor (1989), see also the very simplified but illustrative discussion of Cuerno *et al* (1992).

To summarize, the Poincaré section method may be very illustrative because we can visually determine the chaotic region (i.e. the set of initial conditions that lead to chaotic motion). The limitation of the Poincaré method is its qualitative nature. For quantifying chaos in a Hamiltonian system we need the Liapunov exponents.

4.2. Liapunov exponents

The hallmark of a chaotic system is the sensitive dependence on the initial conditions. This means that if we follow the evolution of two originally close trajectories, and evaluate their separation at any time

$$d_t(x, w) = \sqrt{\sum_{i=1}^{2N} w_i(t)^2}, \tag{8}$$

where $x \equiv (q, p)$, and w is the infinitesimal separation vector in phase space (figure 5), we expect to find

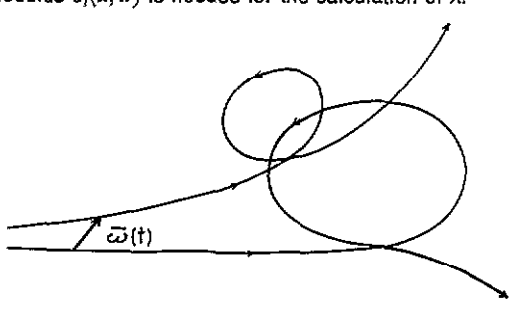
$$d_t(x, w) \propto \exp(\lambda t), \tag{9}$$

λ is clearly the rate of the exponential separation growth in the direction w away from the original trajectory. In a chaotic case λ is a positive number, while in a regular one λ vanishes. This number is one of the most usual indicators of the dynamical behaviour, it is called the Liapunov exponent, and describes how sensitive the system is to perturbations in the direction of w. With expression (9) in mind, it is not difficult to accept that the Liapunov exponent can be evaluated as

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log\left(\frac{d_t(x, w)}{d_0(x, w)}\right). \tag{10}$$

Before we go further, notice two very important points: first, we have $2N$ exponents, since this is the number of independent directions in which we can 'perturb' the initial condition in phase space, and second, the existence of an exponential divergence is only an absolute guarantee of irregular motion when the phase-space region accessible to the system is bounded. For unbounded or scattering states, we can use Liapunov exponents, and even Poincaré sections, for pinpointing possible stochastic motions,

Figure 5. Two initially very close trajectories in phase space illustrating the separation vector $w(t)$, whose modulus $d_t(x, w)$ is needed for the calculation of λ.



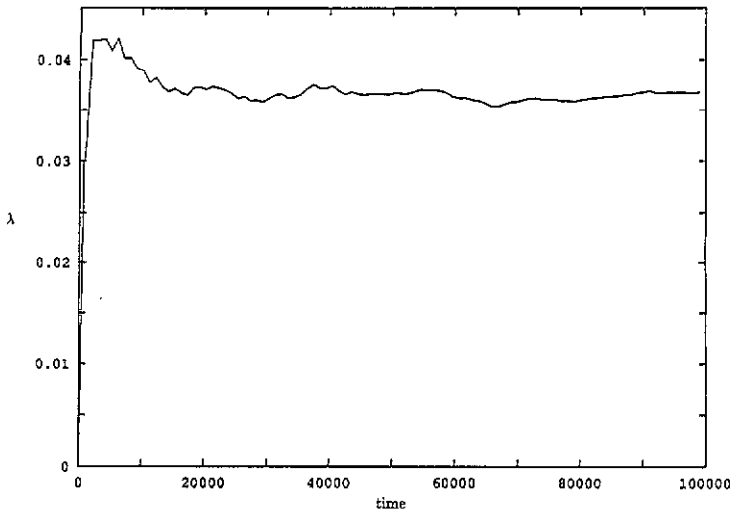


Figure 6. Estimate of the maximum Liapunov exponent $\lambda_{(1)}$ versus integration time, for the chaotic trajectory depicted in figure 4(b). Notice the convergence of the estimates towards $\simeq 0.037$.

but they have to be defined in a different way, see Bercovich *et al* (1991) for some ideas and references. The important point here is that once we have established the positivity of at least one Liapunov exponent in a bounded system, it is definitively chaotic.

We must point out also that the direct application of expression (10), always produces the same number irrespective of w (for details on how to use (10), see Lichtenberg and Lieberman 1983 or Cuerno *et al* 1992). This is so since the growth produced by the largest exponent will quickly overwhelm that caused by the others. To calculate the complete spectrum of $2N$ Liapunov exponents one has to be careful and use special methods (Wolf *et al* 1985).

The maximum Liapunov exponent is enough for determining the type of motion one has for a given set of initial conditions. In the case of a two-degrees-of-freedom system, like the extensible pendulum, the maximum Liapunov exponent determines all exponents. We prove this in what follows.

To begin with, we know that the phase-space volume is preserved since we are dealing with a Hamiltonian system. Let us say that $V(t)$ is the phase-space volume at any time t, then $V(t) = V_0$ where V_0 is the initial volume. According to (9), the volume must evolve as

$$\begin{aligned} V(t) &= V_0 \exp(\lambda_{(1)}t)\exp(\lambda_{(2)}t)\cdots\exp(\lambda_{(2N)}t) \\ &= V_0 \exp[(\lambda_{(1)} + \lambda_{(2)} + \cdots + \lambda_{(2N)})t] \\ &= V_0, \end{aligned} \tag{11}$$

hence, the sum of all Liapunov exponents in a Hamiltonian system vanishes. Furthermore, one of the Liapunov exponents always vanishes since, as no trajectory can diverge from itself, the Liapunov exponent in the direction of the motion must vanish. We can also prove that the Liapunov exponents occur in pairs with the same magnitude but opposite sign, i.e. for any $\lambda_{(i)}$ there exists another $\lambda_{(j)}$, with $i \neq j$, such that $\lambda_{(i)} = -\lambda_{(j)}$. The proof is, in fact, identical to that given above for the vanishing of the sum of all Liapunov exponents, but we have to use the invariance of the area on the two-dimensional subspaces $(q_i - p_i)$, i.e. the so-called Poincaré integral invariants (Sudarshan and Mukunda 1974), instead of the constancy of V. These considerations establish that in a two-degrees-of-freedom system, the maximum Liapunov exponent, $\lambda_{(1)}$, suffices to determine the whole spectrum, since according to the recent discussion, we should have $\lambda_{(4)} = -\lambda_{(1)}$ and $\lambda_{(2)} = \lambda_{(3)} = 0$.

A way to check the qualitative conclusions obtained in the previous section on the chaoticity of the extensible pendulum is by computing the maximum Liapunov exponent λ_1. We have evaluated it for one trajectory in the sea of chaotic points for $c = 0.75$ and $E = 0.04875$; the results of this calculation are exhibited in figure 6, where we clearly see that $\lambda_1 \sim 0.037 > 0$. The trajectory is definitively chaotic.

The simultaneous existence of contracting and expanding directions in a chaotic system, that is of pairs of 'zero-sum' Liapunov exponents, is illustrated in figure 7. This figure exhibits, for the purposes of comparison, the evolution of a small square of initial conditions ($0.01 \leq q_1, p_1 \leq 0.05$) on the Poincaré plane, for regular (figure 7(a)), and stochastic (figure 7(b)) trajectories of the extensible pendulum. In the regular region the separation between close initial conditions changes slowly if at all and the initial square does not suffer great changes or deformations during the evolution. But,

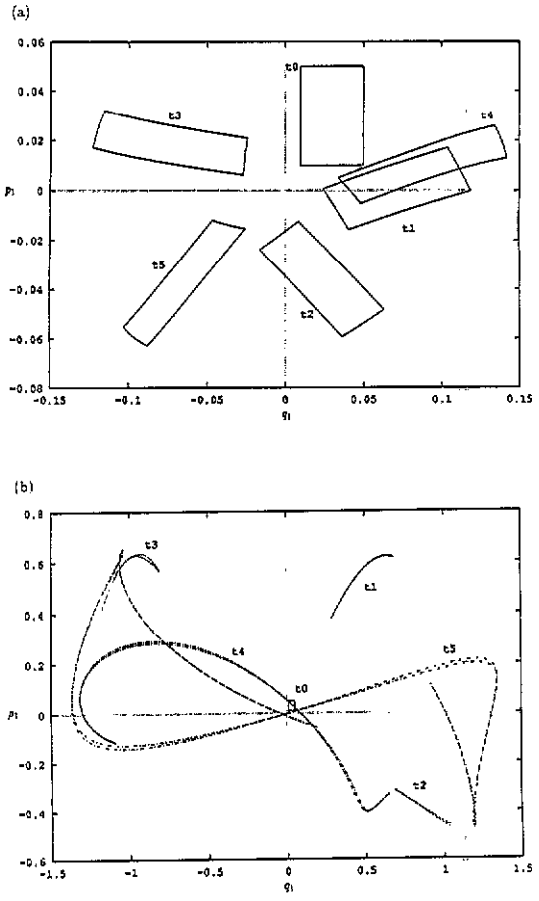


Figure 7. Effect of the dynamics, shown at times $t_0 = 0$, $t_1 = 2.5$, $t_2 = 5$, $t_3 = 10$, $t_4 = 15$, and $t_5 = 20$, on a small area of initial conditions in the Poincaré plane: (a) regular region; (b) stochastic region. The energies are as in figure 4. Notice that the size of the initial square is the *same* in both cases.

in the stochastic region, two antagonistic trends are competing. Close initial conditions separate exponentially with time while, conversely, they must preserve the initial area. This may appear at first sight impossible, but as figure 7(b) exhibits, this is not so, since together with the expansion there is an obvious process of contraction in another direction. The intricacies of the phase-space motion in the chaotic case may at first defy the imagination.

4.3. Correlation function

Another way of exhibiting the difference between a chaotic and a regular orbit is visualizing the 'loss' of memory typical of chaotic evolution, which can be spotted through the *autocorrelation function* (AF) of a trajectory which measures the 'similarity' between a certain initial motion and later ones. In the case of chaotic behaviour, the system quickly loses information about its previous states and the AF therefore tends to zero after a sufficiently long time interval. On the other hand, in a regular trajectory the AF does not vanish. This may be interpreted in the following way: if we integrate the equations of motion of the system from t_0 to t_1, with $t_0 < t_1$, and afterwards we take the final state (t_1) as the initial condition and integrate backwards in time, if the orbit is chaotic we would not be able to come back to the original initial condition at t_0; whereas a regular orbit does not lose information of its past history, and we would be able to return, integrating backwards in time, to the original initial state.

As we have said, the correlation function for a chaotic orbit tends to zero with time because the information of its past does not allow complete confidence in forecasting its future, while the correlation function of a regular orbit oscillates but never tends to zero; this means that it is feasible to predict confidently the future behaviour of the system. These two cases are illustrated in figures 8(a) and (b) for regular and chaotic orbits, respectively, of the extensible pendulum (we used $x = q_1$ in these calculations). It is interesting to notice that the correlation function of a regular orbit decreases and then increases, but always maintains an approximately constant average value. This is so because a periodic or conditionally periodic orbit moves away from its initial condition (the correlation function decreases on the average) but after a certain period of time it comes back near the initial condition (the correlation function increases to approximately the original average value). In the chaotic case, on the contrary, the correlation function vanishes rapidly because the trajectory loses information about its past evolution; however, since the chaotic orbits fill densely the phase space available to the system, there exists a finite time in which the orbit is as close as we may want to the initial condition. This explains why the correlation function of a chaotic orbit (figure 8(b)) grows again after tending to zero and in general oscillates.

5. Conclusions

We have seen that the extensible pendulum is an interesting and seemingly simple Hamiltonian system which displays very complex behaviour. We used numerical techniques to establish the coexistence of chaotic and regular orbits on the phase space of the system. The extensible pendulum is well suited for illustrating the differences between regularity and stochasticity in a system trajectory. To exhibit such different features of the motion we have used as indicators Poincaré mappings, the appearance of the physical orbits, the evolution of a small set of initial conditions on the Poincaré plane, the maximum Liapunov exponent and the autocorrelation function. The simultaneous use of all these methods seems advisable due to the complementary value

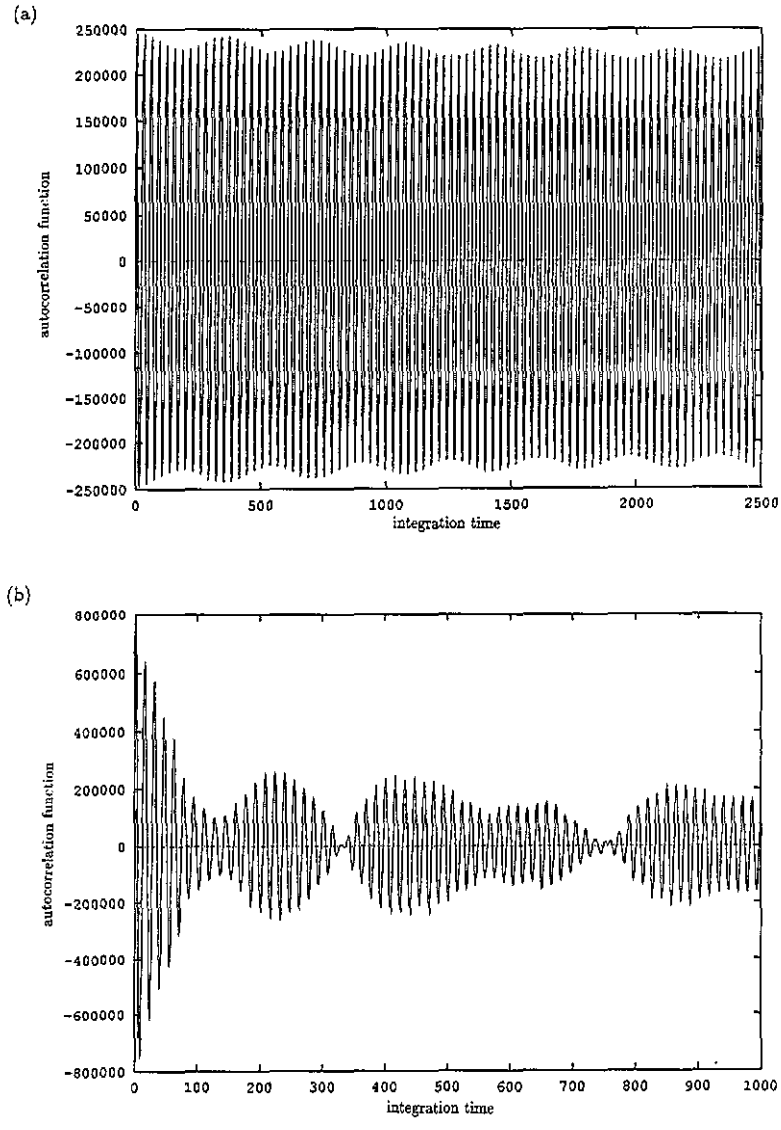


Figure 8. Autocorrelation functions for trajectories in (a) regular region; (b) chaotic region. Energies as in figures 4 and 7. The autocorrelation function was evaluated using the coordinate $q_1(t)$ by taking the Fourier transform of its power spectrum.

they have. We have tried to explain and justify the meaning and some basic properties of the less familiar indicators. Given the apparent simplicity of the extensible pendulum and the great deal of physical and mathematical research done upon it, the system can be useful in mechanics courses to illustrate basic concepts and techniques of nonlinear dynamics. The indicators studied have very direct interpretations and may help to convey the idea that most systems are non-integrable and therefore that they possess very complicated patterns of evolution and, thus, their long term evolution is unpredictable.

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