THE CURVATURE CRITERION AND THE DYNAMICS OF A ROLLING ELASTIC CYLINDER

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ABSTRACT

The motion of an expanding and contracting cylinder rolling on a curved surface is analysed. We establish the existence of autoparametric instabilities for certain values of the cylinder parameters and also found its integrable cases. We next use what we have called the curvature criterion to predict an order-chaos-order transitions whenever the system is chaotic. The Poincaré surface of section method is used to check some of the predictions with good results. The curvature criterion and some of its implications are also briefly discussed.

1. Introduction

An elastic cylinder rolling without slipping on a cylindrical surface may be thought of as a simplistic model for an elastic roller bearing. But, despite its possible uses as a model for something, we show here that it can be a system worth of study by its own sake (Wilms and Cohen 1991, Arizmendi *et al.* 1995). We exhibit that the rolling elastic cylinder is describable using a two-degree-of-freedom Hamiltonian which becomes integrable at two different values of the system parameters. We also show that the system is prone to unstable behaviour at certain values of its parrameters and, after applying what we call the *curvature criterion for the regularity of motion* or CC, we guess that for other parameter values it may exhibit ch ic motions. Furthermore, we predict that whenever the rolling elastic cylinder undergo chaotic oscillations it would also exhibit an order-chaos-order transition on increasing its energy. We argue that this behaviour may be understood as the result of geometric properties of the potential energy surface (PES) of the system. These predictions are checked against numerical computations of the cylinder behaviour. Whenever possible, we use the criterion to predict other features of the Hamiltonian transition to chaos.

2. The rolling elastic cylinder

Let us consider a thin elastic cylinder of radius $r_0 + r$ and mass m, rolling on a cylindrical surface of radius R (Fig. 1). The cylinder is expanding and contracting but never loses its circular profile. We call r_0 the undeformed constant radius and r the *change* in the radius as the oscillation proceeds. The normal v_n and the tangential v_t components of the velocity of point P in the plane of the paper (see Fig. 1) can be easily seen to be

$$\begin{aligned} v_n &= \dot{r} - (r_0 + r)\Omega \sin\phi - \dot{r}\cos\phi, \\ v_t &= \dot{r}\sin\phi + (R - r_0 - r)\dot{\theta} - (r_0 + r)\Omega\cos\phi. \end{aligned} \tag{1}$$

In these equations $\Omega = \dot{\phi}$ is the angular velocity of the cylinder, and the angles θ and ϕ are illustrated in Fig. 1. Furthermore, the rolling without slipping condition gives the relationship $(R - r_0 - r)\dot{\theta} = (r_0 + r)\Omega$ between the variables of the system. With the help of these equations we can get the lagrangian of the system

$$L = m\left(\dot{r}^2 + (r - r_0 - r)^2\dot{\theta}^2\right) - mg\left(R - r_0 - (R - r_0 - r)\cos\theta\right) + \frac{1}{2}kr^2; \tag{2}$$

where g is the acceleration of gravity, m the mass and k the elastic constant of the cylinder. It then follows that the Hamiltonian of the elastic cylinder is

$$H = \frac{1}{4m}\left(p_r^2 + \frac{p_\theta^2}{(a - r)^2} + mga(1 - \cos\theta)\right) + \frac{k}{2}r^2, \tag{3}$$

where we defined $a \equiv R - r_0$.

For the purposes of this work, it is convenient to introduce the dimensionless cartesian coordinates

$$y = (1 - \frac{r}{a}) \cos\theta, \quad \text{and} \quad x = (1 - \frac{r}{a}) \sin\theta, \tag{4}$$

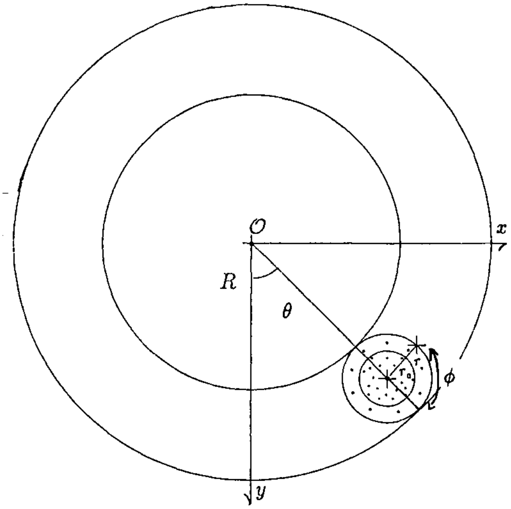

Figure 1. Schematic representation of the system.

for the description of the system. The origin of these coordinates (marked O in Fig. 1) is located at the geometrical centre of the outer cylindrical surface. The problem suggest polar coordinates, but cartesian coordinates are best suited for the numerical calculations since these reduce to normal mode coordinates in absence of the geometrical coupling. These coordinates are also important for the application of CC. The transformation to the new variables x and y can be easily extended to a canonical transformation, in this way we get the dimensionless Hamiltonian function

$$H = \frac{1}{4}(p_x^2 + p_y^2) + c(1 - y) + \frac{1}{2}\left(\sqrt{x^2 + y^2} - 1\right)^2, \tag{5}$$

where we introduced the nonnegative, nondimensional number $c \equiv mg/ka$ parametrizing the behaviour of the system. This number c is the ratio of the weight of the cylinder to the elastic force caused by a deformation a, or may be taken as the ratio of the frequency of rolling back and forth in a pendulum-like fashion on the outer cylindrical surface to the frequency of elastic oscillations of the cylinder. The introduction of this nondimensional H, is equivalent to saying that we are scaling all energies by ka^2 and the times by $\sqrt{m/k}$.

Using the Hamiltonian (5), the equations of motion of the system can be expressed as the canonical system of equations for the phase space coordinates (q_1, q_2, p_1, p_2) of the cylinder

$$\begin{aligned} \dot{x} &= p_x/2, \quad \dot{p}_x = -x + \frac{x}{\sqrt{x^2+y^2}}, \\ \dot{y} &= p_y/2, \quad \dot{p}_y = c - y + \frac{y}{\sqrt{x^2+y^2}}. \end{aligned} \tag{6}$$

Notice that the special case of an elastic cylinder rolling on a flat surface corresponds to taking $c = 0$. The system is integrable in this particular case, since, when $c = 0$, the system becomes equivalent to a central field problem in the plane (Arizmendi *et al.* 1995). Furthermore, the system has two points of equilibrium located at $(0, c+1)$ and $(0, c-1)$, the first point corresponds to stable equilibrium whereas the second one corresponds to a non-stable equilibrium point, notice also the presence of a mild type of singularity at the origin. This singularity is only important when the cylinder fits closely into the external cylindrical surface; the most important consequence of the singularity is that the system becomes again integrable (this is equivalent to taking the $c \to \infty$ limit). This directly follows as the cylinder may only spin about its own axis in such a case. As we see in section 4 when we apply CC to the system, the shape of the potential energy function $V(x, y)$ carries important clues on the behavior of the rolling elastic cylinder. But for now, let us check on the low energy behaviour of it.

3. Parametric resonance

At low enough energies the Hamiltonian (5) can be approximated by

$$H \simeq \frac{1}{4}(p_x^2 + p_y^2) + c(1-y) + \frac{1}{2}\left(\frac{cx^2}{(1+c)} + y^2\right) + \frac{x^2 y}{6(1+c)^2}. \tag{7}$$

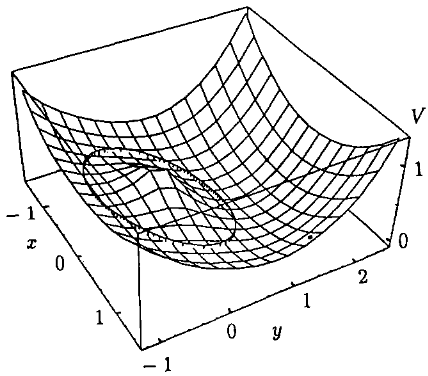

Figure 2. Potential energy surface of the rolling elastic cylinder, the "barrier" marks the zero curvature line.

From this hamiltonian the low energy equations of motion for the cylinder are

$$\ddot{x} + \left(\frac{c}{2(1+c)} + \frac{y}{6(1+c)^2}\right) x = 0, \tag{8}$$

$$\ddot{y} + \frac{y}{2} = -\frac{x^2}{12(1+c)^2}, \tag{9}$$

equation (8) describes a driven harmonic oscillator whereas equation (9) corresponds to a parametric oscillator. If we now assume that $x \ll y$, then we can approximately solve equation (9) with the simple solution $y(t) = A\cos(\omega_2 t)$. Substituting this solution in (8) we get

$$\frac{d^2x}{d\tau^2} + (b + 16p\cos(2\tau))\,x = 0, \tag{10}$$

where we defined $b \equiv 4c/(1+c)$, $12p \equiv A/(1+c)^2$, and $\tau \equiv t/2^{3/2}$. Notice that, with these identifications, we have $0 \leq b \leq 4$. Equation (10) has the standard form of a Mathieu equation and, therefore, to study the low energy behaviour of the system we can use the well known properties of it (McLachlan 1947). The stability properties of the Mathieu equation are summarized in the stability diagram of Fig. 3. As is well known, the solutions to Mathieu equation show unstable behavior at $b = n^2$, $n = 1, 2, 3, \ldots$, but, to evaluate the parameter values at which one should expect the onset of autoparametric oscillations in the rolling elastic cylinder, we have to plot the behavior of it in the same stability chart. This is easily done since we can get for the cylinder

$$p = \frac{A}{192}(4-b)^2. \tag{11}$$

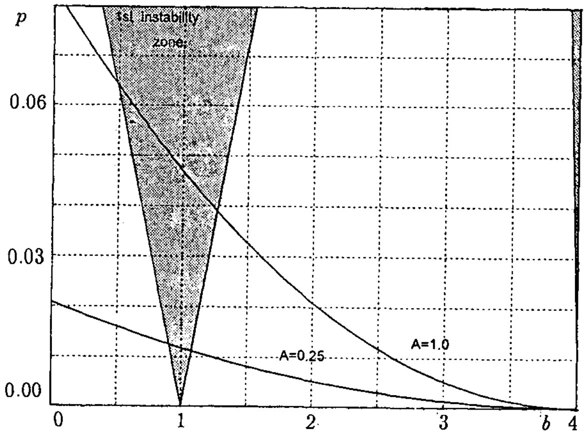

Figure 3. Stability diagram of the Mathieu equation with the behaviour of the cylinder at two A values superimposed.

This curve is plotted for two different A values in figure 3, notice that it always passes trough the point (4,0). On the other hand, since b cannot be greater than four, all instability zones starting at $b = n^2$ for $n \geq 2$ are forbidden to the cylinder, excepting for the point (4,0) — which corresponds to $c = \infty$ and hence to an integrable case— which just touches the apex of the second instability zone.

In this way, we have established that the cylinder may become parametrically unstable when $b = 1$, that is when $c = 1/3$. This is the only parameter value producing the onset of autoparametric oscillations.

4. CC and the cylinder dynamics

A criterion for the appearance of stochastic or chaotic behavior in Hamiltonian systems with two degrees of freedom, like the one under analysis, was suggested by Toda (1974) and later rediscovered by Brumer and Duff (1976); this was a variant of what we have called CC. This criterion was used for addressing problems in several fields (Bolotin *et al.* 1989, Núñez–Yépez 1990, Núñez–Yépez *et al.* 1990, Akhiezer *et al.* 1991, Carretero–González 1992) but it does not appear to be reliable enough (Núñez–Yépez *et al.* 1990) and thus, it is not much used now. We have recently analysed the matter and have found that this criterion is in fact reliable, but for studying the regularity, *not the chaotic*, properties of Hamiltonian systems (Arizmendi *et al.* 1995). CC is easy to describe; to analyse whether a two-dimensional Hamiltonian system shows only regular motions or not, it suffices to compute the Gaussian curvature, K, of the potential energy surface (PES). *If K is always greater than zero then the system cannot be chaotic.* This assertion is justified in the Appendix. The Gaussian curvature of a two-surface $z = V(x, y)$ is defined as ($V_{ij} \equiv \partial^2 V/\partial x_i \partial x_j$)

$$K = \frac{V_{xx}V_{yy} - V_{xy}^2}{\left(1 + V_{xx}^2 + V_{yy}^2\right)^2}, \tag{12}$$

K is the function we have to analyse for deciding whether a system is regular or not. If $K \geq 0$, like in the case of an harmonic oscillator, the system is got to be regular. The system under analysis has the possibility of being chaotic, only if there is a region with negative curvature on its PES. Obviously, there is no guarantee that the system is going to behave chaotically but, if the chaotic behaviour can be established by other means, then, we may use CC to predict other properties and to obtain estimates of energies pertaining to the transition. For example, if the PES of a chaotic system has got a confined and bounded region of negative curvature, at large energies the system would spend most of its time moving outside that region and the motion would be then regular again. In other words, *a chaotic system with a confined and bounded PES always shows an order-chaos-order transition as its energy is increased.* We may even use these ideas to estimate the energy value at which the transition to chaotic motion takes place. To this end we have to compute the set of points for which $K(x, y) = 0$, and determine the minimum energy, E_c, necessary for reaching this curve. E_c would play the role of the critical energy for the onset of chaos since, for any $E > E_c$, the phase trajectory of the system is able to get to the negative curvature region and the motion may thus become chaotic. We may also estimate the energy of maximum chaos (*i.e.* the energy at which the volume of phase space occupied by chaotic orbits has a maximum), E_{max}, as the maximum energy the system attains on the zero curvature line. It should be clear that this is only a rather poor lower bound estimate, as it is based on the assumption that at energies greater than E_{max} the phase-trajectories spend very little time traversing through the $K < 0$ region (Núñez-Yépez 1990, Carretero-González 1992).

To analyse the rolling elastic cylinder from the perspective of the CC we have to pay attention to the potential energy function (7). A plot of the PES of the cylinder is shown in Fig. 3. Notice that $-c^2/2$ is the minimum energy the cylinder may have. To check for the possible existence of chaos in this system we have to evaluate if there is a region of negative curvature in its PES. This is easily done since K is

$$K = \frac{(x^2+y^2)^{\frac{3}{2}} - (x^2+y^2)}{(x^2+y^2)^{3/2}} \tag{13}$$

which take negative values inside a circle of radius 1 centered at the origin (Fig. 2), thus we may expect chaotic behaviour. As Fig. 4 shows the system is indeed chaotic for $c \neq 0$. The existence of chaos already established, we may estimate the critical energy for the onset of chaos as the minimun energy for reaching the zero curvature line: $E_c = 0$. Thus if $c \neq 0$, for energies $E > 0$ there must exist phase space points traversed by chaotic orbits. Furthermore, as the negative curvature region is confined, the criterion predicts that at large enough energies the motions become regular again. The energy of maximum chaos can also

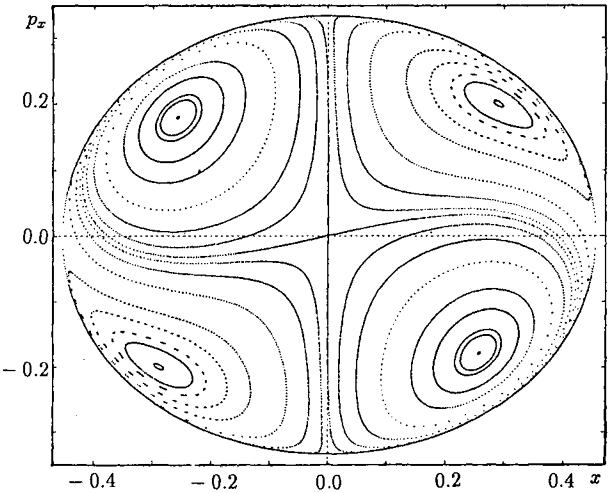

Figure 4a. Poincaré section for $c = 1/3$ at $E = -1/36$. Notice the regularity of the motions.

be easily estimated, it is $E_{max} = c$.

To test these conclusions, we have numerically integrated the canonical equations (6) and plotted the points where the phase orbit pierces the plane x-p_x when $p_y > 0$, $y = 1 + c$. In this way we construct Poincaré sections for several energies at $c = 1/3$. The onset of chaos as the energy is increased is apparent in Fig. 4. Note the existence of only regular orbits at negative energies and the presence of an increasingly noticeable chaotic splatter of points near the origin as the energy of the system grows beyond zero. Notice also the obvious inversion symmetry in the Poincaré sections, which reflects the fact that the cylinder oscillates back and forth in the course of its rolling motion but that it does not have enough energy for going around circling completelly the outer cylindrical surface. Also, as predicted by CC, chaotic motions disappear at energies greater than a certain energy $E_c' > E_{max}$. The breaking of the aforementioned symmetry

Figure 4b. Poincaré section for $c = 1/3$ at $E = 1/64$. As predicted by CC, chaos begin to appear.

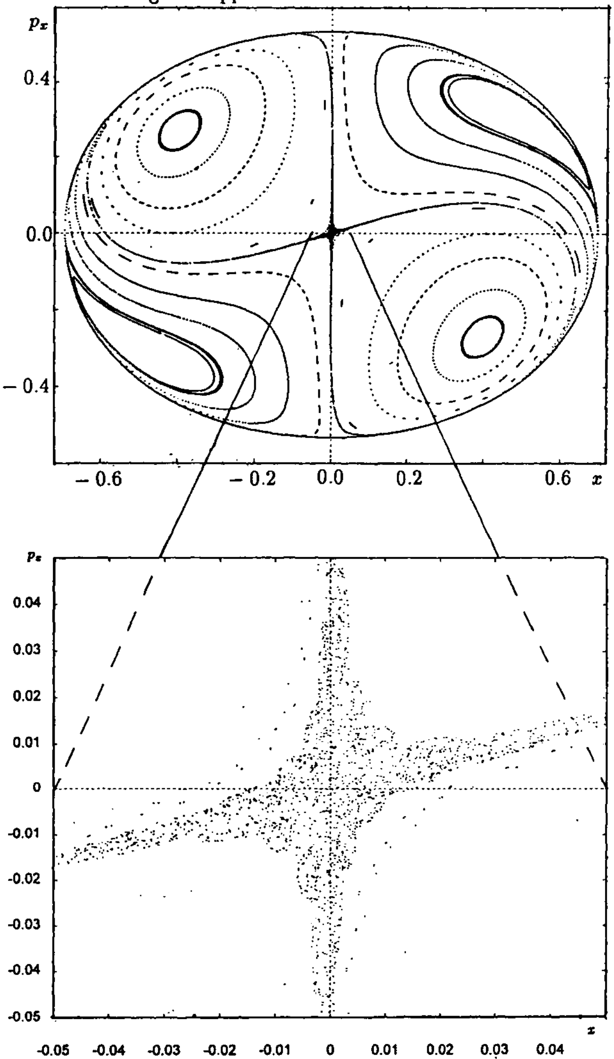

Figure 4c. Poincaré section for $c = 1/3$ at $E = 1/64$. Blow up of the chaotic region in 4b.

is also apparent in the plots at higher energies where the motion becomes regular again. This breakdown is related to the ocurrence of rotational motions in the cylinder which is thus capable of going in a preferred direction, either clockwise or counterclockwise, on the inner surface of the outer cylinder. This makes clear the fact that the chaotic oscillations of the cylinder are a result of the interaction of its radial elastic vibrations with the oscillations produced in the course of its rolling motion. The predictions we get from applying CC are completelly fulfilled by the numerically calculated dynamics of the system, but the quantitative agreement is so to speak fortituous. As you may check, at other c-values, $E_c = 0$ is only a lower bound estimate, the actual transition to chaos occurs at a higher energy. But the important point is, as long as the system is chaotic, an order-chaos-order transition is always found to occur.

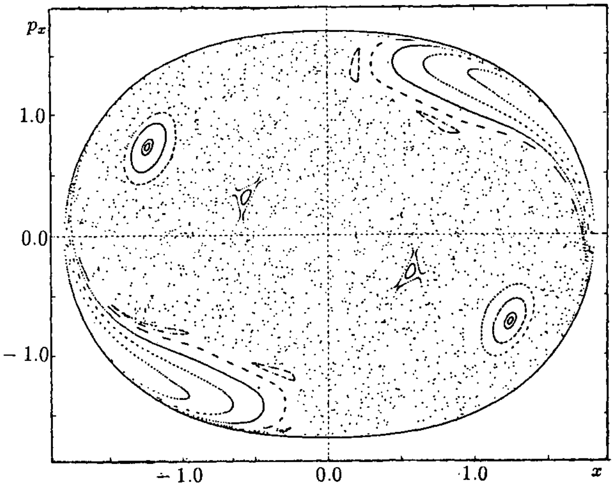

Figure 4d. Poincaré section for $c = 1/3$ at $E = 2/3$.

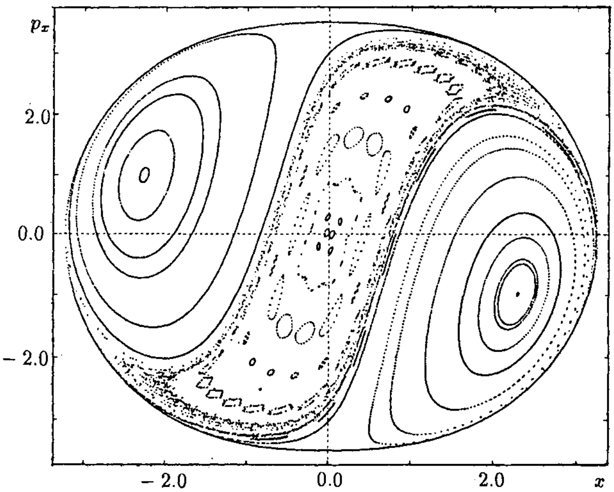

Figure 4e. Poincaré section for $c = 1/3$ at $E = 3$. The motions have become regular again.

5. Appendix. The curvature criterion and the order-chaos-order transition

To understand the mechanism behind what we have called the curvature criterion (Toda 1974, Núñez–Yépez *et al.* 1990, Akhiezer *et al.* 1991), let us consider two initially very close trajectories $\mathbf{z}(t) \equiv (x, y, p_x, p_y)$ and $\mathbf{z}'(t) = \mathbf{z}(t) + \boldsymbol{\xi}(t)$ in the phase-space of a two-dimensional Hamiltonian system. This means that, if we write the phase space distance between the two trajectories $\mathbf{z}$ and $\mathbf{z}'$ as

$$||\xi(t)|| = d_t(\mathbf{z}', \mathbf{z}) \equiv \sqrt{\sum_{a=1}^{2}(x'_a(t) - x_a(t))^2 + (p'_a(t) - p_a(t))^2}, \qquad (A1)$$

then, assuming that the system is chaotic, this distance should change as

$$d_t = d_0 \exp(\lambda t), \qquad (A2)$$

where λ is a number, essentially a Liapunov exponent (Carretero-González *et al.* 1994), characterising the rate of exponential divergence at $t = 0$. It should be clear now that the vector $\boldsymbol{\xi}$ lives in the tangent space to the phase space at the point $\mathbf{z}(t)$. As we are interested in local properties of the motion, we have to analyse the Jacobi or variational equations for the deviations $\boldsymbol{\xi}$ —which may be obtained in an elegant way from a variational principle (Salas–Brito 1984). The Jacobi equations can be written as the non-autonomous linear system

$$\frac{d}{dt}\boldsymbol{\xi} = \mathcal{M}(t) \cdot \boldsymbol{\xi}, \qquad (A3)$$

where the matrix $\mathcal{M}(t)$ is the product of the symplectic matrix (Arnold 1978)

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad (A4)$$

times the matrix of second derivatives of H (Cf. equation (5)) evaluated along the fiducial trajectory $\mathbf{z}(t)$:

$$\mathcal{M}(t) \equiv \left.\begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ -H_{xx} & -H_{xy} & 0 & 0 \\ -H_{yx} & -H_{yy} & 0 & 0 \end{pmatrix}\right|_{\mathbf{z}(t)}, \qquad (A5)$$

where $H_{ij} \equiv \partial^2 H/\partial x^i \partial x^j$. The matrix $\mathcal{M}(t)$ can be transformed to a diagonal form with the help of a similarity transformation. Such transformation is crucial since the eigenvalues λ of this matrix characterize the nature of the motion. If one of these eigenvalues is real, then the trajectories diverge exponentially. But if λ is imaginary, the trajectories just oscillate one around the other and thus the motion is regular. Generally, however, the eigenvalues of $\mathcal{M}(t)$ changes with time; but we can make the approximation, good for small t and thus compatible with the local caracterization we are trying to obtain, of replacing the time dependent point on the trajectory $\mathbf{z}(t)$ by the phase space *coordinates* $\mathbf{z}$. With this approximation the problem is greatly simplified, the four eigenvalues of the now constant matrix $\mathcal{M}$ are also constant and are given by

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left(-1 + (1 - \frac{4\alpha}{\beta^2})^{1/2}\right)^{1/2}, \\ \lambda_2 &= \frac{1}{2}\left(-1 - (1 - \frac{4\alpha}{\beta^2})^{1/2}\right)^{1/2}, \\ \lambda_3 &= -\frac{1}{2}\left(-1 + (1 - \frac{4\alpha}{\beta^2})^{1/2}\right)^{1/2}, \\ \lambda_4 &= -\frac{1}{2}\left(-1 - (1 - \frac{4\alpha}{\beta^2})^{1/2}\right)^{1/2}, \end{aligned} \tag{A6}$$

where, for the sake of simplicity, we introduced $\alpha \equiv V_{xx}V_{yy} - V_{xy}^2$ and $\beta \equiv V_{xx} + V_{yy}$ and $V(x, y)$ is the potential energy function appearing in (5). Notice that the eigenvalues appear in pairs with the opposite sign and thus that their sum always vanishes: $\sum_{a=1}^{4} \lambda_a = 0$. These are expected properties in any 2-D Hamiltonian system like the cylinder under analysis (Lichtenberg and Lieberman 1992, Carretero–González *et al.* 1994).

A straightforward analysis of the eigenvalues (A6) show that they are real— that is, the system is chaotic— when $\alpha < 0$. Hence $\alpha < 0$ is a necessary consequence of the chaoticity of the system. The criterion can then be stated in a more easy to remember form by saying that if a two-dimensional Hamiltonian system is chaotic then there necessarily exists a zone of its potential energy surface (PES) where the Gaussian curvature is negative (Toda 1974, Brumer and Duff 1976, Bolotin *et al.* 1989, Núñez–Yépez *et al.* 1990). This easilly follows since, as the Gaussian curvature K of a two-surface $z = V(x, y)$ is defined as in equation (12), the sign of K obviously coincides with that of α and, hence, with the condition for real eigenvalues. There is thus a relationship between chaos producing instabilities and the sign of the Gaussian curvature of the PES of the system, which is what we have called CC (Akhiezer *et al.* 1991). Notice that the criterion only says that if the system is chaotic then its PES should have a negative curvature region. The just stated criterion cannot predict if a given two-dimensional Hamiltonian system is chaotic or not.

However, CC does ascertain that a system with no negative-curvature region in its PES has always to be regular, since this is just the converse of the CC statement. As we have explained in section 4 if the motions of a system are known to be chaotic, CC would then be able to predict if chaos can be made to dissapear on further increasing the energy. That is, CC can be used used to predict whether a chaotic system shall exhibit an order-chaos-order transition or not. For this to happen, it is only necessary that the negative curvature region of the system's PES be confined to a bounded region in configuration space; for, in such a case, at large enough energies the system would spend most of its time moving in a positive curvature region and thus the motion would become regular again (Bolotin *et al.* 1989, Arizmendi *et al.* 1995). This is basically what happens with the rolling elastic cylinder potential-energy function. Thence, as the system is chaotic for almost all c values, we can ascertain the existence of order-chaos-order transitions in all the chaotic cases.

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