VARIATIONAL APPROXIMATIONS FOR A NONLOCAL DISCRETE NLS EQUATION

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Abstract: Applying variational methods, we construct analytical approximations for one-peak discrete solitons supported by a nonlocal discrete nonlinear Schrödinger equation describing laser beams in nematic liquid crystal waveguides. Two variational ansatz approximations are used: one uses the exact long range decay from the linearized solution and the other uses the decay as a variational parameter. Comparisons with numerical solutions are presented.

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1 INTRODUCTION

We construct analytical approximations for discrete solitons of a nonlocal discrete nonlinear Schrödinger equation (DNLS) with a cubic Hartree-type nonlinearity introduced by Fratalocchi and Assanto [5] to study the propagation of laser beams in waveguide arrays built from a nematic liquid crystal substratum. To obtain these approximations we extend the variational principles used in Ref. [3] and [4] for 1D discrete solitons in the cubic-quintic DNLS equation.

Breathers solutions that minimize the Hamiltonian over configurations with fixed l^2 norm were studied in Ref. [2]. These solutions are even and decay monotonically away from its maximum. We propose an ansatz with exponential decay to obtain analytical approximations to the breather solutions. Two types of approximations are provided, the first one is based on the asymptotic decay of the soliton's tail, which is obtained from the exact decay of the linearized problem, and the approximation of the amplitude using a variational principle. This approximation retains the shape of the exact breather solution.

The second approximation is provided by solving the Euler-Lagrange equations associated with the effective lagrangian obtained from an ansatz that includes the decay rate as a variational parameter. This reduced system depends on two parameters: the amplitude and the decay, and provides a better approximation for the sites close to the peak at the expense of losing the exact asymptotic decay of the soliton.

These analytic approximations are compared with the one-peak numerical solutions with exponential decay found in Ref. [1]. One goal of the variational method is that it predicts that in the case of the cubic Hartre-type nonlinearity treated here there are no bifurcation phenomena as it do occurs in the cubic-quintic DNLS (cf. Ref. [3]).

2 NONLOCAL DNLS EQUATION

Consider the one-dimensional nonlocal DNLS equation

$$\dot{u}_n = \delta i \left(u_{n+1} + u_{n-1} - 2u_n\right) + 2\gamma \tanh \frac{\kappa}{2} i \sum_{m\in\mathbb{Z}} e^{-\kappa|m-n|} \left|u_m\right|^2 u_n, \qquad n \in \mathbb{Z},\ t > 0. \tag{1}$$

where $\delta,\ \gamma$ are real and $\kappa > 0$. [1] Equation (1) can be formally written as the Hamiltonian system

[1]The physically relevant case is $\delta\gamma > 0$. In the examples analysed below we will consider $\delta, \gamma < 0$.

$$\dot{u}_n = -i\frac{\partial H}{\partial u_n^*}, \qquad n \in \mathbb{Z}. \tag{2}$$

with

$$H = \delta \sum_{n\in\mathbb{Z}} |u_{n+1} - u_n|^2 - \gamma \tanh\frac{\kappa}{2} \sum_{n,m\in\mathbb{Z}} |u_m|^2 e^{-\kappa|m-n|} |u_n|^2 . \tag{3}$$

Let as seek a breather solutions of Eq. (1) by substituting the ansatz $v_n = A_n e^{-i\omega t}$ (namely: all the sites oscillate synchronously). Then, the stationary lattice field A_n must solve the recurrence equation

$$-\omega A_n = \delta(A_{n+1} + A_{n-1} - 2A_n) + 2\gamma \tanh\frac{\kappa}{2} \sum_{m\in\mathbb{Z}} e^{-\kappa|m-n|} |A_m|^2 A_n, \qquad n \in \mathbb{Z}. \tag{4}$$

This stationary equation can be derived from the lagrangian

$$L = \sum_{n\in\mathbb{Z}} \omega |A_n|^2 - \delta \sum_{n\in\mathbb{Z}} |A_{n+1} - A_n|^2 + \gamma \tanh\frac{\kappa}{2} \sum_{n,m\in\mathbb{Z}} |A_m|^2 e^{-\kappa|m-n|} |A_n|^2 . \tag{5}$$

To aproximate breathers solutions to the Eq. (1) we propose the ansatz with exponential decay given by $A_n = Ae^{-\alpha|n|}$ with $\alpha > 0$. Replacing this ansatz on the the lagrangian (5) we obtain the effective lagrangian

$$L_{\text{eff}}(A,\alpha) = A^2 \left((\omega - 2\delta)\coth\alpha + 2\delta \operatorname{csch}\alpha\right) + \tag{6}$$

$$\gamma A^4 \tanh\frac{\kappa}{2} \frac{\coth\left(\frac{\kappa}{2} + \alpha\right)\sinh 2\alpha - \coth 2\alpha \sinh\kappa}{\cosh 2\alpha - \cosh\kappa}. \tag{7}$$

Analytical approximations to the exact soliton solutions to Eq. (5) can be obtained by solving the Euler-Lagrange equations associated to this effective lagrangian. We follow two paths to approximate solutions with exponential decay. In Sec. 2.1 we use a value of α determined by the linearized solution to Eq. (4). While in Sec. 2.2 we solve the Euler-Lagrange system for both parameters A and α in Eq. (6).

2.1 DECAYING TAIL APPROXIMATION

To obtain the decaying tail of the soliton we can view Eq. (4) as a recurrence relation between consecutive amplitudes and consider its linearization which corresponds to the linear Schrödinger equation

$$\begin{cases} A_{n+1} = \left(2 - \frac{\omega}{\delta}\right) A_n - B_n, \\ B_{n+1} = A_n. \end{cases} \tag{8}$$

The eigenvalues associated with the matrix of the system (8) are

$$\lambda_1 = \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - 1} \quad \text{and} \quad \lambda_2 = \frac{1}{\lambda_1}, \tag{9}$$

where $a = 2 - \omega/\delta$. Considering the manifold $A_n = A\lambda^{-|n|} = Ae^{-\alpha|n|}, \forall n \in \mathbb{Z}$, we obtain

$$\alpha = \ln\left(\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - 1}\right). \tag{10}$$

We observe from Eq. (9) that a necessary condition for the existence of a soliton solution is $\omega\delta < 0$. The value of α given in Eq. (10) provides the correct asymptotic decay of the soliton and it will be more accurate as the nonlinearity is smaller. On the other hand, the amplitude A can be found as a variational parameter from the Euler-Lagrange equation obtained from Eq. (6), $\frac{\partial L_{\text{eff}}}{\partial A} = 0$, which neglecting the trivial solution $A = 0$, yields the following quadratic equation:

$$(\omega - 2\delta)\coth\alpha + 2\delta\,\mathrm{csch}\,\alpha + A^2 2\gamma\tanh\frac{\kappa}{2}\frac{\coth\left(\frac{\kappa}{2}+\alpha\right)\sinh 2\alpha - \coth 2\alpha\sinh\kappa}{\cosh 2\alpha - \cosh\kappa} = 0. \quad (11)$$

Under the above condition $\omega\delta < 0$, it is straightforward to check that the quadratic equation (11) has a solution if $\gamma\delta > 0$, which correspond to the physically relevant cases.

In Fig. 1 the comparison between the numerically exact solutions (black triangles) and the approximation with the linearization used above (red circles) recovers the shape of the soliton and it is more accurate when the nonlocality is smaller, i.e. κ is larger.

2.2 Global approximation

We now consider A and α as variational parameters and we seek their values from the Euler-Lagrange system obtained from the effective lagrangian (6)

$$\frac{\partial L_{\text{eff}}}{\partial A} = 0 \quad , \quad \frac{\partial L_{\text{eff}}}{\partial \alpha} = 0. \quad (12)$$

The first equation is the same computed above in Eq. (11). For the sake of brevity we omit here the explicit formulation of the second equation. The system can be solved numerically. Comparisons depicted in Fig. 1 between these approximations and numerical exact solutions show that they approximate better the solution near the peak than the approximation with asymptotic decay.

It can be checked that, under the same conditions of the previous section, i.e. $\omega\delta < 0$ and $\delta\gamma > 0$, the quadratic equation (11) depending on A has a unique positive root for each $\alpha > 0$. This indicates that we do not expect bistability phenomena for the soliton solutions of the nonlocal DNLS with a cubic Hartree-type nonlinearity, as it is the case in the cubic-quintic DNLS (cf. Ref. [3]).

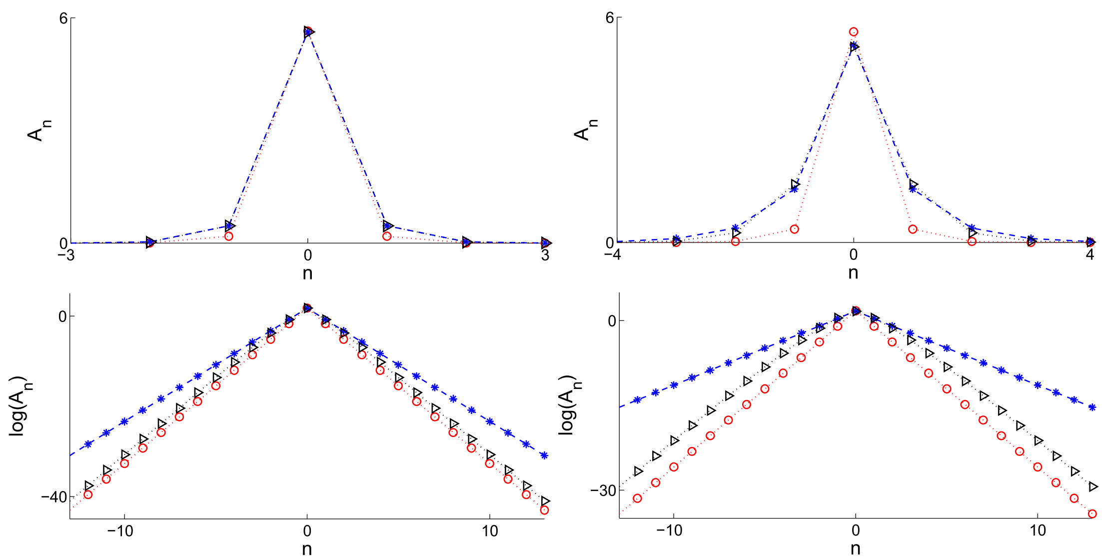

Figure 1: Soliton solutions of Eq. (4) A_n vs. site number (top) and log(A_n) vs. site number (bottom) for $\kappa = 0.5$, $\delta = -0.5$, $\gamma = -1$, $\omega = 14.6749972$ (left) and $\kappa = 0.25$, $\delta = -0.5$, $\gamma = -1$, $w = 6.97912841$ (right). Black triangles correspond to numerically exact solutions, red circles to linearized decay and blue asterics correspond to Euler-Lagrange system approximation for both paramenters A and α.

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