Preface

Wave phenomena is a theme of broad scientific interest. As such, waves are studied in many diverse contexts, ranging from mathematics and physics to engineering, biosciences, chemistry, and finance just to name a few. The relevant phenomenology is typically studied by means of partial differential equations (PDEs), as well as differential-difference equations (DDEs)—also referred to as nonlinear lattices or lattice differential equations.

Of particular interest are *nonlinear waves* and related phenomena, which are described by *nonlinear PDEs and DDEs*, which naturally emerge when the underlying physical mechanisms are nonlinear. Examples where the study of nonlinear waves is relevant, traverse all scales—from atomic to cosmic ones!—including, e.g., Bose–Einstein condensates, the interaction of electromagnetic waves with matter, optical fibers and waveguides, acoustics, water waves, atmospheric and planetary scales, and even galaxy formation. It is thus not surprising that nonlinear waves are of broad significance to mathematicians and scientists of various disciplines alike.

How it all began

Studies on nonlinear waves, and particularly the problem of surface gravity waves in deep water, started already from the nineteenth century by Stokes (1847). Stokes developed the correct form of nonlinear water wave equations, and found traveling periodic wave solutions that have an important feature: their velocity is amplitude-dependent (e.g., larger amplitude waves move faster than the lower amplitude ones), which is a common feature in the case of nonlinear waves. At about the same time, Riemann got interested in nonlinear wave phenomena, and in one of his classic works (1860), he devised an ingenious method to solve a certain system of nonlinear PDEs describing gas dynamics. Riemann's ideas and methodology have been generalized to include one of the most outstanding, genuinely nonlinear, wave structure arising from relevant studies: the *shock wave* (i.e., an emergent discontinuity in the dependent variables of the underlying PDEs).

Although nowadays it is an almost common belief that the modern theory of nonlinear waves actually started with the pioneering works of Stokes and Riemann, the story of nonlinear waves started, in fact, a bit earlier. Indeed, a prominent milestone was the discovery of *solitary waves* by John Scott Russell (1834) who, after reporting his observations in a Scottish channel, he also performed a series of pertinent experiments in a long and narrow water tank. Russell's observations, on a single isolated wave crest that propagates without widening or steepening, were strongly doubted by leading figures in fluid dynamics of that time; Airy (1841), for instance, claimed that the wave was linear, and hence should disperse. Nevertheless, the importance of Russel's findings was finally appreciated. Indeed, at first, Boussinesq, in his work in the 1870s, found new nonlinear equations for shallow water waves, as well as a solitary wave solution of a sech^2-profile, with the amplitude, velocity, and width of the solution being simultaneously controlled by a sole parameter. Two decades later, Korteweg and de Vries (1895) simplified the shallow water wave equation (subsequently called, per their initials, the "KdV equation") and presented a nonlinear periodic solution, nowadays known as "cnoidal wave", with a special limiting case (the infinite period limit) being the solitary wave solution.

A series of fundamental contributions in the field of nonlinear waves came into light in the second half of the last century. Whitham (1965) discovered an asymptotic method ("Whitham modulation theory") for studying slowly varying periodic waves, which can be thought of as a generalization of the WKB (Wentzel–Kramers–Brillouin) theory for nonlinear PDEs. Importantly, Whitham's theory was able to provide an analytical description of the emergence and evolution of *dispersive shock waves*, i.e., nonlinear wave structures that resolve a wave-breaking singularity in cases where dispersion dominates dissipation.

On the other hand, also in 1965, Zabusky and Kruskal found that the KdV equation arises in the continuum limit of a one-dimensional (1D) anharmonic lattice introduced in 1955 by Fermi, Pasta, Ulam, and Tsingou ("FPUT lattice"). The latter was motivated by the question of "thermalization" in solids, i.e., how energy is distributed among the lattice modes. Importantly, in their simulations, Zabusky and Kruskal discovered that an initial condition of the form of a harmonic wave disintegrates into a "train" of solitary waves satisfying the KdV equation. These solitary waves were found to undergo *elastic collisions* whereby each solitary wave remained unscathed, retaining its shape and velocity, after its interaction with other solitary waves, thus featuring particle-like properties. For this reason, Zabusky and Kruskal introduced for the first time the term *soliton*, originating from "solit-ary" and "on", usually used for particles.

Shortly after the work of Zabusky and Kruskal, in their pioneering work, Gardner, Greene, Kruskal, and Miura (1967, 1974) developed an "inverse method" to solve the initial value problem for the KdV equation supplemented with rapidly decaying initial data. The key idea of this method is to associate the KdV with the eigenvalue problem of the linear Schrödinger equation and then, employing concepts and methods of direct and inverse scattering, determine the solution of KdV. This was a major achievement, as the inverse method turned out to be an exact "algorithm" determining the solution of a nonlinear PDE (the KdV equation in this case), by solving a *succession* of linear sub-problems. In this sense, the inverse method for the KdV equation was a much more sophisticated "linearization" procedure than that of the linearization of another nonlinear model, the viscous Burgers equation: the latter, was "linearized" by Hopf (1950) and Cole (1951) upon using an ingenious transformation ("Cole–Hopf transformation"), which reduced the viscous Burgers equation to the linear heat equation.

Was the KdV (like the Burgers equation) just a sole, very particular type of model, where the inverse method could be applied? Further developments showed that this was not the case and, in fact, it turned out that a plethora of physically relevant nonlinear evolution equations can be solved via the inverse method. Indeed, Lax (1968) proposed a framework under which numerous PDEs can be represented in terms of two linear operators (their corresponding "Lax pair"). These operators satisfy a compatibility condition—expressed as the commutation of derivatives of Lax operators—which ensures that the time evolution of the Lax pair corresponds to the original PDE. Employing Lax' ideas, a few years later, Zakharov and Shabat (1971) presented a variant of the inverse method, which relies on a matrix form of the Lax pair operators, and derived the soliton solutions of the nonlinear Schrödinger (NLS) equation.

All the above ideas culminated in the work of Ablowitz, Kaup, Newell, and Segur (AKNS) (1974), who developed a *matrix formalism* of the inverse method, which could treat a wide range of nonlinear evolution equations of physical significance; examples include the previously studied models, KdV and NLS, but also the modified KdV and sine–Gordon equations, as well as higher order variants of all the above equations. Importantly, AKNS showed that their scheme—which they termed *Inverse Scattering Transform* (IST)—is in fact a generalization of the Fourier transform method that is used to solve linear PDEs with constant coefficients and, hence, can be viewed as a Fourier transform for nonlinear problems.

... and the Golden Age of nonlinear waves and solitons had just dawned.

The next day(s)

The development of the IST has triggered numerous further investigations. On the one hand, it offered the motivation to study many of the above-mentioned and related equations by functional analytic methods, with the aim of proving local, and whenever possible, global existence of solutions to the relevant initial value problems, a program which continues to this day. On the other hand, following an activity that started in the 1960s and 1970s, several asymptotic methods were developed. These methods allowed for the derivation of effective nonlinear wave equations (such as the KdV or the NLS equations), which were relevant in a wide range of physical contexts. Accordingly, wide classes of equations, including numerous nonlinear PDEs solvable by IST, were identified and solved.

This way, various PDEs (and systems thereof), as well as higher order equations in (1+1)-dimensions, multidimensional models, discrete systems (described by differential–difference and partial difference equations), as well as, more recently, nonlocal and fractional nonlinear PDEs and others, were introduced and analyzed. Out of this activity, envelope solitons, kinks, breathers and discrete breathers, rational solutions (including "rogue waves"), dispersive shock waves—often called "undular bores" in fluid dynamics—vortical structures, and many other species (compactons, peakons, similaritons, and the list goes on and on) of nonlinear waves were and are still discovered and studied. Importantly, this research led to further developments and discoveries of mathematical techniques needed to analyze the relevant problems. Such techniques include direct methods to obtain soliton solutions, perturbation theories for solitons, variational methods relying on the use of conservation laws, as well as of the Hamiltonian or Lagrangian structure associated with these equations, methods for studying the existence and stability of solitons, and many others. Hand-in-hand with these developments, computational methods have emerged (with an ever-growing pace in the past few decades) aiming to complement this analytical understanding and to offer a valuable platform for so-called "in-silico" analysis and experiments.

Physics and, in many cases, also engineering, have played an important role in the tremendous advancements in the theory of nonlinear waves, featuring a "symbiotic"-type of relationship. Indeed, nonlinear waves are inherently connected—featuring continuous developments and mutual benefit—with the fields of water waves and fluid dynamics, acoustics, plasma physics, nonlinear optics, field theory, Bose–Einstein condensation, superfluidity and superconductivity, electrical and mechanical lattices, the study of atmospheric and oceanic systems and even the formation of cosmic structures (e.g., galaxies) and dark matter. On the other hand, a number of important applications, such as long-distance optical fiber communications, photonic and phononic devices, logic gates and metamaterials, laser systems, superconducting devices (e.g., Josephson junctions), medical imaging, vibration control, cloaking, and energy harvesting are just a few prominent examples highlighting the diverse ways in which engineers leverage the unique characteristics of nonlinear waves to enhance the performance and capabilities of various systems and technologies.

The above discussion is an attempt to highlight how the field of nonlinear waves has been shaped during its remarkable development over the past few decades, while aiming at the same time to reveal the fascinating and often unexpected behavior arising in nonlinear systems. The interplay between theory, experiments, and computations has been crucial in advancing our understanding of nonlinear wave phenomena, and many more important and exciting developments are lying ahead, waiting to be explored!

Background and the role of this book

The aim of this book is to provide a self-contained introduction to the continuously developing field of nonlinear waves, that offers the background, the basic ideas and mathematical, as well as computational methods, while also presenting an overview of associated physical applications. The idea for this book originated from our research activity in this field for almost three decades and from the need to have such a book to hand to (and work through with) students curious to explore this fascinating world of nonlinear waves with us. It has been shaped over many years from our lectures in relevant courses, which we taught at San Diego State University, the National and Kapodistrian University of Athens, and the University of Massachusetts at Amherst. Although the primary purpose of this book is to serve as a textbook, the selection and exposition of the material should also be useful to anyone who wishes to be introduced to the field of nonlinear waves. At the same time, we could not resist including some topics closer to our heart and to our own recent research activity in the hope that they may trigger the more seasoned reader's curiosity to explore and advance the subject further!

Our focus is on conservative (Hamiltonian) systems that support nonlinear wave phenomena, so the study of dissipative systems and pertinent effects constitutes only a side (e.g., perturbative) feature herein. Nevertheless, many of the techniques that we develop herein could be used also in the latter class of systems. Importantly, we consider both continuum and discrete systems, with more emphasis being given on the former, as many techniques to analyze discrete systems can be borrowed from the study of the continuum ones. We also make sure to present suites of "inherently discrete" methods stemming from the highly discrete limit. As concerns the various species of nonlinear waves explored in this book, emphasis is given to solitary waves and solitons, but nonlinear periodic waves, shock waves, and dispersive shock waves are also discussed. Despite the extent of our presentation, our intention is not to be exhaustive as concerns the choice of the material covered. Instead, our purpose is to convey to interested readers basic ideas and methodologies, as well as some of the more important results obtained in the field, while inviting (and providing resources for) the reader to delve further into the subject.

For the above reasons, we have made the effort to keep the length of each of the thirty-three chapters of the book limited to approximately ten to fifteen pages. Despite the natural flow of the presentation, each chapter is reasonably self-contained so that, in principle, it may constitute the material of a lecture (or a week of lectures) in a relevant course. Following such an approach, each chapter contains a number of exercises which, on the one hand, could be used for homework problems, and on the other hand, may be considered by the reader as a supplement, motivating a deeper understanding and practice on the material of the chapter, while often bringing them closer to recently addressed research topics. Chapters including a higher level of mathematical sophistication (or rather specialized material) are indicated with an asterisk. Such portions can be safely omitted on a first pass of the book (certainly in an introductory nonlinear waves course for instance) and left for a later, deeper dive for the more curious reader.

Another important feature of the book is the inclusion of numerical techniques, and programs written in Matlab, relevant to the solution of various problems on nonlinear waves. The pertinent chapters focus both on numerical analysis concepts and the outcomes of the relevant computations, which highlight—and make direct connection with—analytical results. We believe that incorporation of this computational part of the book is valuable for a number of reasons. First of all, for scientists and engineers, it is nowadays a common practice to employ numerical methods in order to study a specific nonlinear system; it is thus necessary to provide relevant ideas and tools that are prototypically useful, in our view, for numerically studying problems of nonlinear waves. Second, the use of computational techniques and the derivation of numerical results can prove to be particularly important: on the one hand, one may directly test the validity of analytical approximations against the "exact" numerical result; on the other hand, even in cases where no analytical predictions are available, the outcome of the computations may provide valuable hints for further analytical efforts.

Organization of the presentation

The book is organized in five parts, each containing a number of chapters, which are devoted to the main fundamental models that appear generically in the study of nonlinear waves. The structure is as follows.

- Part I. Introduction and Motivation of Models: three chapters (Chaps. 1–3),
- Part II. Korteweg–de Vries (KdV) Equation: ten chapters (Chaps. 4–13),
- Part III. Klein–Gordon, sine–Gordon, and ϕ^4 models: four chapters (Chaps. 14–17),
- Part IV. Nonlinear Schrödinger Equations: thirteen chapters (Chaps. 18–30),
- Part V. Discrete Models: three chapters (Chaps. 31–33).

In Part I, we set the stage by discussing at first the progression from one-degree-of-freedom problems (i.e., a sole nonlinear oscillator) to ones with many degrees of freedom (a chain of nonlinear oscillators as, e.g., the FPUT lattice). As this path eventually leads to continuum systems (i.e., to nonlinear dispersive PDEs, like the KdV equation), we proceed by introducing important ideas and notions appearing in the context of linear dispersive PDEs (as, e.g., dispersion, wave velocities, and so on). This part of the book is concluded by the introduction of a variety of nonlinear dispersive wave models (that will be analyzed in the subsequent chapters), which appear generically in a wide range of contexts and applications in physics, engineering, etc.

Part II is devoted to the study of the KdV equation and some of its siblings, such as the Boussinesq, KdV Burgers, and Kadomtsev–Petviashvili equations. We show how this model can be derived, from first principles, in various physical systems (e.g., in the FPUT lattice, or in shallow water waves), establish its asymptotic connection with relevant models (e.g., the Boussinesq equation), and present its basic properties and solutions. As KdV is a generic nonlinear dispersive wave equation, we provide insights into the role and the nature of dispersion and nonlinearity mechanisms. In particular, we present asymptotic techniques to study its linearized version and its dispersionless version, namely the Hopf equation, that may give rise to the emergence of shock waves. Solutions of the KdV, including nonlinear periodic waves ("cnoidal waves") and solitons, are derived by means of dynamical system techniques. In addition, we present more sophisticated methods, namely the IST and Bäcklund transformation for the KdV equation, while the notion of self-similarity, and self-similar solutions, is also discussed. Important models belonging to the "KdV family", such as the KdV–Burgers equation (which serves as a prototypical model supporting dispersive shock waves), and the Kadomtsev–Petviashvili equation (2D generalization of the KdV) are also introduced. Furthermore, a direct perturbation theory for solitons is presented in order to analyze the dynamics of KdV solitons under, e.g., dissipative or transverse perturbations.

In Part III, we present another type of models, belonging to the class of nonlinear Klein–Gordon equations, with the sine–Gordon equation and the so-called ϕ^4 model being the most prominent members of this family. After providing a physical motivation and explaining how such models can emerge (e.g., in mechanical lattice systems), we proceed by discussing basic properties and solutions. This way, soliton solutions of these models, including kinks and breathers, are again found by various methods, namely dynamical system techniques and Bäcklund transformations. Soliton stability and interactions are studied by means of various methodologies and relevant tools—such as direct and spectral methods, the Evans function, and the so-called Manton's method for soliton interactions—are introduced and explained in some detail. Furthermore, the asymptotic connection between Klein–Gordon models and the NLS equation is established, leading to the effective description of sine–Gordon or ϕ^4 breathers as NLS solitons.

Next, in Part IV, we study equations of the NLS type, with the most prominent variant being the Gross–Pitaevskii (GP) equation (NLS with an external potential) which is used in the physics of Bose–Einstein condensation. We show how the NLS equation can generically describe wave packets in media with weak dispersion and nonlinearity, we introduce the notion of modulational instability of plane waves, and derive nonlinear periodic waves and soliton solutions by means of dynamical systems techniques. There are two types of NLS solitons, bright and dark ones, for a focusing or a defocusing nonlinearity, respectively. Bright solitons are also derived by means of IST, while small-amplitude dark solitons are shown to behave as KdV solitons, as found by the formal asymptotic NLS to KdV connection. NLS serves as a prominent example, where field-theoretic ideas and methods that we introduce, such as symmetries, conservation laws, and Noether's theorem, can be applied. Conservation laws, as well as the Lagrangian structure of the NLS and GP models, are used to develop the adiabatic perturbation theory, and a variational approach, for bright and dark solitons, which are considered in a number of physical problems. The stability of 1D solitons in the presence of the trap, as well as in the 2D setting, is studied by various techniques. We also analyze multicomponent versions of the NLS, and discuss the properties and dynamics of the quintessential 2D structures occurring in the defocusing 2D setting, namely vortices.

The final part of the book, Part V, is devoted to the study of various discrete models, including Klein–Gordon, discrete NLS, Ablowitz–Ladik, and the Toda lattice. We analyze the effect of discreteness on solitary waves in dynamical lattices, introducing a number of notions, ideas, and techniques. Accordingly, we discuss the so-called anti-continuum limit of vanishing coupling, which can be used to identify the stability of the states that can be directly continued from this limit, and then complement this perspective with a near-continuum, energy-based one, showcasing the famous Peierls–Nabarro barrier. We thus show that continuum solitons have two discrete "incarnations", a site-centered and an intersite-centered (or bond-centered) one, featuring different stability properties. We discuss possibilities of "exceptional discretizations" (suitably crafted to conserve discrete analogues of conserved quantities), and also present results for higher dimensional settings, and for integrable nonlinear dynamical lattices, such as the Ablowitz–Ladik model and the Toda lattice.

It should be added that, interspersed throughout the book, we include chapters presenting the numerical methods accompanying and complementing the corresponding analysis for each class of models.

We recognize that there are numerous additional topics of recent and ongoing research excitement, including dispersive shock waves, non-Hermitian nonlinear media, the interplay of topology and nonlinearity, as well as fractional and more generally nonlocal nonlinear media. The topics of this book are not expected to be exhaustive and, motivated by the excitement of these ongoing directions and others at a nascent stage (such as the impact of data-driven techniques in nonlinear waves), we can already envision a future amendment of the present volume. In the meantime, we still hope that the present contribution will be a helpful introductory point of reference for junior as well as hopefully also for more seasoned researchers in this field.

It is our genuine hope that this book will serve as an invitation for the reader to dive deeper into the magic of Nonlinear Waves and use this text as a springboard for adding new chapters to this fascinating subject. Happy reading!