

Contents

PART I INTRODUCTION AND MOTIVATION OF MODELS

1	Introduction and Motivation	3
1.1	Few degrees of freedom	3
1.2	Many degrees of freedom	11
1.3	A nonlinear variant: the FPUT lattice	13
	Exercises	15
2	Linear Dispersive Wave Equations	17
2.1	Dispersion relations and relevant notions	17
2.2	Examples of linear dispersive wave equations	19
2.3	Wavepackets and group velocity	23
2.4	Dissipation, instability, and diffusion	25
	Exercises	27
3	Nonlinear Dispersive Wave Equations	29
3.1	Dispersion relations, linear and nonlinear equations	29
3.2	Unidirectional propagation: KdV, KP, and NLS	29
3.3	Bidirectional propagation: KG and Boussinesq	36
	Exercises	40

PART II KORTEWEG–DE VRIES (KDV) EQUATION

4	The Korteweg–de Vries (KdV) Equation	45
4.1	Obtaining KdV as a limit of FPUT	45
4.2	Obtaining KdV for shallow water waves	48
4.3	The effects of dispersion and nonlinearity	51
4.4	Putting it all together: the Zabusky–Kruskal numerical experiments	61
4.5	A cute twist: conservation laws	63
	Exercises	65
5	From Boussinesq to KdV – Boussinesq Solitons as KdV Solitons	68
5.1	Boussinesq to a single KdV (right-going waves)	68
5.2	Boussinesq to a KdV pair (right- and left-going waves)	70
5.3	Connecting the Boussinesq soliton with the KdV soliton	72
5.4	A higher dimensional generalization: Boussinesq to KP	74
	Exercises	75

6	Traveling Wave Reduction, Elliptic Functions, and Connections to KdV	77
6.1	Traveling wave reduction: the quadrature problem	77
6.2	Elliptic function solutions	80
6.3	Another traveling wave reduction example: the Boussinesq equation	84
	Exercises	85
7	Burgers and KdV–Burgers Equations – Regularized Shock Waves	90
7.1	Shock waves of the Burgers equation	91
7.2	The Cole–Hopf transformation	94
7.3	The KdVB equation – a model for dispersive shocks	100
	Exercises	105
8	A Final Touch From KdV: Invariances and Self-Similar Solutions	107
8.1	Scale and Galilean invariances	107
8.2	The KdV self-similar waveforms	110
8.3	Going beyond the KdV: seeking self-similarity	113
8.4	A familiar example with a little-known twist: the diffusion equation	114
	Exercises	115
9	Spectral Methods	117
9.1	Revisiting the Fourier transform and the golden property	117
9.2	Spectral methods for PDEs	118
	Exercises	125
10	Bäcklund Transformation for the KdV	126
10.1	The general idea and a simple introductory example: Laplace equation	126
10.2	Bäcklund transformation for the KdV	128
	Exercises	132
11	Inverse Scattering Transform I – the KdV Equation*	134
11.1	Introduction and motivation	134
11.2	Description of the Inverse Scattering Transform method	135
11.3	The linear problem associated with the KdV equation	140
11.4	The problem of the evolution of the scattering data	142
11.5	The inverse problem – Gel’fand–Levitan–Marchenko (GLM) equation	147
11.6	Soliton solutions of the KdV equation	149
	Exercises	156
12	Direct Perturbation Theory for Solitons*	157
12.1	General aspects of the perturbative approach	157
12.2	KdV with a dissipative perturbation	160
12.3	Transverse instability of line solitons of the KP equation	166
	Exercises	171
13	The Kadomtsev–Petviashvili Equation*	172
13.1	The KP equation and its variants	172
13.2	Basic properties of the KP equation	174
13.3	Line solitons and lumps of the KP equation	175

13.4 Interactions of line solitons of the KP-II equation	177
13.5 Finite-genus and quasi-periodic solutions	182
Exercises	187

PART III KLEIN–GORDON, SINE–GORDON, AND PHI-4 MODELS

14 Another Class of Models: Nonlinear Klein–Gordon Equations	191
14.1 Models, physical motivation, and principal kink waveforms	191
14.2 Linear and nonlinear transformations	195
14.3 Linear stability, homogeneous steady states, linear dispersion relations	199
14.4 Linearization around non-homogeneous states: kinks	201
14.5 A stability tool: the Evans function	203
Exercises	209
15 Additional Tools/Results for Klein–Gordon Equations	210
15.1 Finding solitons using the method of Bäcklund transforms	210
15.2 Examining soliton interactions: the Manton method	212
15.3 Kink-kink and kink-antikink collisions	214
15.4 Higher dimensional sine-Gordon equation: a teaser and an invitation	219
15.5 Sine-Gordon with gain and loss: a cute application	221
Exercises	223
16 Klein–Gordon to NLS Connection – Breathers as NLS Solitons	225
16.1 Preliminaries: model and the method of multiple scales	225
16.2 NLS from Klein–Gordon	227
16.3 Sine-Gordon breathers as NLS bright solitons	229
16.4 On the formal derivation and universality of NLS	232
Exercises	234
17 Interlude: Numerical Considerations for Nonlinear Wave Equations	235
17.1 Existence and stability	235
17.2 Nonlinear dynamics	240
Exercises	245

PART IV THE NONLINEAR SCHRÖDINGER EQUATIONS

18 The Nonlinear Schrödinger (NLS) Equation	249
18.1 Obtaining linear and nonlinear Schrödinger from dispersive wavepackets	249
18.2 Plane wave solutions and modulational instability	252
18.3 A more general analysis: solitons and periodic solutions	256
Exercises	264
19 NLS to KdV Connection – Dark Solitons as KdV Solitons	267
19.1 Preliminaries and motivation: structure of the dark soliton	267
19.2 Defocusing NLS to Boussinesq and to KdV	270

19.3	Direct derivation of the KdV	274
19.4	Shallow dark solitons vs. KdV solitons	276
	Exercises	278
20	Actions, Symmetries, Conservation Laws, Noether's Theorem, and All That	280
20.1	Lagrangian and Hamiltonian formalisms	280
20.2	Poisson brackets	285
20.3	Noether's theorem	287
20.4	Field-theoretic variants	288
20.5	The nonlinear Schrödinger case example	291
20.6	Connections with the cubic NLS and its soliton linearization problem	294
20.7	Connections with wave collapse: NLS with generalized power law nonlinearity	296
20.8	Final twist: antisymmetric operators and generalized Poisson brackets	299
	Exercises	300
21	Applications of Conservation Laws – Adiabatic Perturbation Method	303
21.1	Scaling and phase invariances and Galilean boost for NLS solutions	303
21.2	Conservation laws and conserved quantities for NLS: redux	305
21.3	Adiabatic perturbation theory for bright solitons	307
21.4	Adiabatic perturbation theory for dark solitons	312
	Exercises	319
22	Numerical Techniques for NLS	322
22.1	Finite differences	322
22.2	Steady states: Newton's method	324
22.3	Stability	329
22.4	Dynamics: finite differences and RK4	331
22.5	Example: soliton on top of a hill	334
22.6	Validating your codes and results	336
	Exercises	337
23	Inverse Scattering Transform II – the NLS Equation*	338
23.1	The Ablowitz–Kaup–Newell–Segur (AKNS) approach	338
23.2	IST for the focusing NLS equation	344
23.3	Soliton solutions	352
23.4	A note on the defocusing NLS and the role of boundary conditions	358
23.5	The inverse problem as a Riemann–Hilbert problem	358
	Exercises	361
24	The Gross–Pitaevskii (GP) Equation	362
24.1	Physical motivation: Bose–Einstein condensates	362
24.2	The Gross–Pitaevskii (GP) equation	363
24.3	Dimensional reductions and adimensionalization	364
24.4	Small and large mass limits	368
	Exercises	370
25	Variational Approximation for the NLS and GP Equations	372

25.1 Preliminaries	372
25.2 Illustrative examples: the NLS solitons	375
25.3 Perturbation theory within the VA approach	380
Exercises	392
26 Stability Analysis in 1D	397
26.1 Linear stability analysis for the GP equation	397
26.2 Time-independent problem	398
26.3 Time-dependent problem	402
26.4 Application to the linearization problem	406
Exercises	411
27 Multi-Component Systems	413
27.1 Dark-bright solitons in multi-component systems	413
27.2 Twist I: a dark (trapping a) bright soliton	415
27.3 Twist II: dark-dark soliton in the integrable limit	415
27.4 Twist III: dark-bright solitons for general couplings	417
27.5 Twist IV: 2-DB-soliton solution in integrable limit and beyond	418
27.6 Twist V: lattices of solitons with general couplings	419
27.7 Twist VI: dark-bright solitons in a trap	421
Exercises	423
28 Transverse Instability of Solitons Stripes – Perturbative Approach	424
28.1 Transverse instability of bright solitons	424
28.2 Transverse instability of dark solitons	430
28.3 From the 2D defocusing NLS to KP	438
Exercises	441
29 Transverse Instability of Dark Stripes – Adiabatic Invariant Approach	442
29.1 Adiabatic invariants	442
29.2 The case of dark soliton stripes	444
29.3 The case of ring dark solitons	447
Exercises	449
30 Vortices in the 2D Defocusing NLS	452
30.1 Vortex solutions	452
30.2 Vortex dynamics in inhomogeneous backgrounds	455
30.3 NLS vortices as classical point-vortices	459
30.4 Generating functions for vortex interactions	463
Exercises	466
 PART V DISCRETE MODELS	
31 The Discrete Klein–Gordon Model	471
31.1 Discrete nonlinear Klein–Gordon models	471
31.2 A complementary perspective: energy landscape	477
31.3 Exceptional discretizations: a first glance via momentum conservation	482

31.4	Exceptional discretizations: a second glance via Bogomol'nyi bounds	484
31.5	Exceptional discretizations: a final unifying view	487
31.6	Integrable discretizations: the sine-Gordon case	488
	Exercises	491
32	Discrete Models of the Nonlinear Schrödinger Type	492
32.1	DNLS: existence and stability analysis	492
32.2	Integrable discretizations: the Ablowitz–Ladik model	506
	Exercises	509
33	From Toda to FPUT and Beyond	512
33.1	Toda lattice: an integrable starting point	512
33.2	Going beyond Toda: FPUT variants	517
	Exercises	523
	References	524
	Index	535