Supplement: An Adiabatic Invariant Approach to Transverse Instability: Landau Dynamics of Soliton Filaments

P. G. Kevrekidis,[1] Wenlong Wang,[2] R. Carretero-González,[3] and D. J. Frantzeskakis[4]

[1]*Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01003-4515 USA*
[2]*Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843-4242, USA*
[3]*Nonlinear Dynamical Systems Group,[*] Computational Sciences Research Center, and Department of Mathematics and Statistics, San Diego State University, San Diego, California 92182-7720, USA*
[4]*Department of Physics, National and Kapodistrian University of Athens, Panepistimiopolis, Zografos, 15784 Athens, Greece*

I. DERIVATION OF EQUATIONS FOR THE DARK SOLITON STRIPE.

Firstly, we provide here a proof of the dynamics for a dark soliton stripe [Eq. (7) in the main text] starting from the corresponding adiabatic invariant Hamiltonian. The conservation of this Hamiltonian leads to

$$\frac{dE}{dt} = \frac{4}{3}\int_{-\infty}^{\infty} \frac{\partial}{\partial t}\left[\left(1+\frac{{x_0}_y^2}{2}\right)\left(\mu - V(x_0) - {x_0}_t^2\right)^{3/2}\right] dy = 0. \tag{S.1}$$

Using the notation

$$\begin{aligned} A &\equiv \mu - V(x_0) - {x_0}_t^2, \\ B &\equiv 1 + \frac{{x_0}_y^2}{2}, \end{aligned}$$

the integrand I of the above integral reads

$$I = x_{0y}x_{0yt}A^{3/2} + \frac{3}{2}A^{1/2}B\left(-V'(x_0)x_{0t} - 2x_{0t}x_{0tt}\right).$$

Now, if we integrate by parts the first term in the integral, assuming boundary conditions such that surface terms disappear, we obtain

$$\int_{-\infty}^{\infty} x_{0y}x_{0yt}A^{3/2}dy = -\int_{-\infty}^{\infty} x_{0t}\left(x_{0y}A^{3/2}\right)_y dy.$$

Now all the terms in the integral multiply x_{0t} and hence in order for the integral to vanish for arbitrary choices of x_0, the term multiplying x_{0t} should vanish. In fact, this is rather intuitive because the reverse process (i.e., multiplying the resulting equation by x_{0t} and integrating over y) leads naturally to the conservation law for the energy. This way, we obtain the final PDE for the filament's dynamical evolution:

$$x_{0tt}B + \frac{1}{3}x_{0yy}A = x_{0y}x_{0t}x_{0yt} - \frac{1}{2}V'(x_0)\left(B - {x_0}_y^2\right). \tag{S.2}$$

Equation (S.2) has a number of meaningful limits:

- Near the linear limit, as indicated in the main text, it leads to

$$x_{0tt} + \frac{\mu}{3}x_{0yy} = 0, \tag{S.3}$$

  yielding the proper linear growth rate of the transverse instability [1].

- If $x_0 = x_0(t)$ only, it recovers the celebrated result for the single dark soliton (obtained originally by Busch and Anglin [2] and also by Konotop and Pitaevskii [3]),

- In the case where there is no potential, i.e., for $V(x_0) = 0$, it yields Eq. (7) in the main text, namely:

$$x_{0tt}B + \frac{1}{3}x_{0yy}(\mu - {x_0}_t^2) = x_{0y}x_{0t}x_{0yt}. \tag{S.4}$$

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II. DERIVATION OF EQUATIONS FOR THE RING DARK SOLITON.

We proceed in a similar fashion for the derivation of the equation of motion in the case of the ring dark soliton. Using the notation

$$C \equiv \mu - V(R) - R_t^2,$$
$$D \equiv 1 + \frac{R_\theta^2}{2R^2},$$

we obtain that:

$$\frac{3}{4}\frac{dE}{dt} = \int_0^{2\pi} \left[R_t D C^{3/2} + \frac{R_\theta R_{\theta t}}{R} C^{3/2} - \frac{R_\theta^2}{R^2} R_t C^{3/2} + \frac{3}{2} R D C^{1/2} (-V'(R) - 2R_{tt}) R_t \right] d\theta = 0. \tag{S.5}$$

Following a similar pattern as the derivation in the previous section, we recognize that all other terms multiply R_t except for the second one, for which we use integration by parts as follows:

$$\int_0^{2\pi} \frac{R_\theta R_{\theta t}}{R} C^{3/2} d\theta = -\int_0^{2\pi} R_t \frac{\partial}{\partial \theta} \left(\frac{R_\theta}{R} C^{3/2} \right) d\theta.$$

Here, the surface terms disappear due to the periodic boundary conditions in the angular variable. As a result, once again all terms multiply R_t and hence this multiplicative prefactor of R_t in the angular integral has to vanish for the integral to generically vanish. The resulting PDE for the ring evolution reads:

$$CD - \frac{R_{\theta\theta}}{R} C = -\frac{R_\theta}{R} \left(\frac{3}{2} V'(R) R_\theta + 3 R_t R_{t\theta} \right) + RD \left(\frac{3}{2} V'(R) + 3R_{tt} \right) \tag{S.6}$$

Again a series of special limits are of particular value:

- For the homogeneous steady state (time- and angle-independent θ) R_0, we obtain $\mu - V(R_0) = \frac{3}{2} R_0 V'(R_0)$, in line, e.g., with Ref. [4].
- In the case of angle-independent radial motion, we obtain

$$R_{tt} = \frac{C}{3R} - \frac{1}{2} V'(R), \tag{S.7}$$

  in agreement with Ref. [5].
- Finally, assuming $R = R_0 + \epsilon R_1(t) \cos(n\theta)$, we obtain to $\mathrm{O}(\epsilon)$ the evolution for R_1 in the form:

$$R_{1tt} = \frac{V'(R_0)}{3R_0} \left[-\frac{5}{2} + \frac{3}{2} n^2 - \frac{3}{2} \frac{R_0 V''(R_0)}{V'(R_0)} \right] R_1. \tag{S.8}$$

  Assuming $R_1 \sim e^{i\omega t}$, we obtain the modes of vibration

$$\omega^2 = \frac{V'(R_0)}{2R_0} \left[\frac{5}{3} - n^2 + \frac{R_0 V''(R_0)}{V'(R_0)} \right] \tag{S.9}$$

  In the case of a parabolic trap, $V(r) = \frac{1}{2}\Omega^2 r^2$, this leads to $\omega = \Omega((8/3 - n^2)/2)^{1/2}$. These constitute Eqs. (10) and (11) in the main text.

[1] E. A. Kuznetsov and S. K. Turitsyn, Zh. Eksp. Teor. Fiz. **94**, 119 (1988) [Sov. Phys. JETP **67**, 1583 (1988)].
[2] Th. Busch and J. R. Anglin Phys. Rev. Lett. **84**, 2298 (2000).
[3] V. V. Konotop and L. P. Pitaevskii, Phys. Rev. Lett. **93**, 240403 (2004).
[4] W. Wang, P.G. Kevrekidis, R. Carretero-González, D.J. Frantzeskakis, T.J. Kaper, and M. Ma Phys. Rev. A **92**, 033611 (2015).
[5] A. M. Kamchatnov and S. V. Korneev, Phys. Lett. A **374**, 4625 (2010).