Faraday waves in Bose-Einstein condensates

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Motivated by recent experiments on Faraday waves in Bose-Einstein condensates we investigate both analytically and numerically the dynamics of cigar-shaped Bose-condensed gases subject to periodic modulation of the strength of the transverse confinement. We offer a fully analytical explanation of the observed parametric resonance, based on a Mathieu-type analysis of the non-polynomial Schrödinger equation. The theoretical prediction for the pattern periodicity versus the driving frequency is directly compared to the experimental data, yielding good qualitative and quantitative agreement between the two. These results are corroborated by direct numerical simulations of both the one-dimensional non-polynomial Schrödinger equation and of the fully three-dimensional Gross-Pitaevskii equation.

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I. INTRODUCTION

Pattern formation in driven systems is an important direction of current research that influences many fields ranging from hydrodynamics to biophysics and from nonlinear optics to reaction kinetics; see [1] for a comprehensive review of the topic.

Some of the oldest and most well-known forms of such phenomena are the so-called Faraday patterns, stemming from the classical studies of Faraday in 1831 [2], who studied the behavior of “groups of particles [placed] upon vibrating elastic surfaces” and (in the Appendix of his much-celebrated paper) the dynamics of “fluids in contact with vibrating surfaces.” Faraday’s experiment became a classical example of pattern formation, whereby the uniform state loses its stability against spatially modulated wave forms, whose dominant length scale is determined by the intrinsic properties of the system (such as the dominant wavelength of the instability) and is only weakly dependent on boundary or initial conditions.

In the past few years, there has been an increase in literature about observing phenomena of the above type in driven superfluids. Bose-Einstein condensates [3,4] offer perhaps the ideal playground for such experiments, since their experimental tunability permits one to create such parametric resonance phenomena in a multiplicity of ways. One such way is by driving the magnetic trap confining the system, as was proposed in Ref. [5]. In the same spirit of modulating the confinement of the system to observe parametric resonances, the works of Refs. [6,7], motivated by the experiments of Ref. [8], considered a periodic modulation of an optical lattice potential; on the other hand, the later work of Ref. [9] focused on modulating in time the confinement potential in a toroidal trap. Another recent suggestion was to produce a parametric drive by means of periodically modulating the scattering length [10], which is directly related to the prefactor of the effective nonlinearity (due to the mean-field interparticle interactions) of the system. This can be achieved by means of Feshbach resonances [11].

These theoretical propositions motivated the very recent experimental implementation of the Faraday waves in Bose-Einstein condensates in the work of Ref. [12]. The actual realization of the spatially modulated patterns arose in a somewhat different way than was proposed in the above studies, a way which is very close, however, to the spirit of the theoretical suggestion of Ref. [13] (see also the more recent consideration of a similar problem from a quantum point of view in Ref. [14]). In particular, in Ref. [12], an elongated cigar-shaped condensate was used where the transverse, strong confinement directions were periodically modulated in time, while the weaker longitudinal direction confinement was time independent. The parametric excitation at the driving frequency was recognized as being responsible for exciting oscillations at half the driving frequency, which is the main resonance also observed in Faraday’s experiments. Subsequently, based on this insight and the dispersion relation of longitudinal collective modes presented in Refs. [15,16], a relation was derived (and convincingly compared to the experimental results in a quantitative manner) for the resulting pattern periodicity versus the transverse driving frequency.

The aim of the present paper is to provide a complete analysis of the instability from first principles and to obtain a fully analytical prediction that can be used for a detailed comparison with the experimental results and numerical results obtained from the one-dimensional (1D) model reduction as well as full three-dimensional (3D) simulations at the mean-field level.

The principal feature, which allows us to provide a detailed quantitative analysis of the system, is the fact that for cigar-shaped condensates (such as the ones used in the experiment of Ref. [12]) there is a quantitatively accurate description in the form of the nonpolynomial Schrödinger equation (NPSE) derived in Ref. [17] (provided that the transverse direction stays close to its ground state, which is

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approximately the case in the experiments of Ref. [12]). The remarkable feature of the NPSE is that in the resulting partial differential equation (PDE) for the longitudinal description of the condensate, the transverse frequency enters explicitly and hence provides, in this setting, the explicit parametric drive that will lead to the observed spatially modulated patterns. To elucidate this feature, we will perform a linear stability analysis of the spatially uniform states in this effective one-dimensional setting. This naturally leads to a Mathieu equation. Then, one can use the theory of the Mathieu equation to identify the most unstable mode and the wave number of the spatial periodicity. This, in turn, allows us to obtain a fully analytical expression for the pattern periodicity as a function of the driving frequency that can be directly compared to the experiment. We believe that this approach yields both qualitative and quantitative insight on the experiment and on the nature of the relevant phenomena.

Our presentation will be structured as follows. In Sec. II, we develop our analytical approach and directly compare it to the experimental findings. In Sec. III, we complement our theoretical findings with numerical results simulating the full 3D experimental setting and comparing it with results of its 1D analog, namely, the NPSE equation. Finally, in Sec. IV, we summarize our findings and present some interesting directions for future work.

II. ANALYTICAL CONSIDERATIONS

Our starting point will be the standard mean-field model for Bose-Einstein condensates (BEC), namely, the three-dimensional Gross-Pitaevskii (GP) equation [3,4]

$$i\hbar \frac{\partial \psi}{\partial t} = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + gN|\psi|^2 \right] \psi,$$  \hspace{1cm} (1)

where $\psi$ denotes the BEC wave function (normalized to unity), $N$ represents the number of atoms, and $g = 4\pi\hbar^2a_s/m$ is proportional to the scattering length $a_s$ of the interatomic interactions; the external potential confining the atoms is given by

$$V(r,z) = \frac{1}{2}m\omega_r^2r^2 + U(z),$$

where $r^2 = x^2 + y^2$, $\omega_r$ is the transverse trapping frequency, and $U(z)$, in the case of Ref. [12], represents a far weaker parabolic, longitudinal, potential (with a magnetic trap frequency less than 5% of the transverse frequency), which will therefore be neglected in the analytical treatment of this section.

Following the approach of Ref. [17], the three-dimensional wave function can be decomposed into a radial and a longitudinal part as

$$\psi(r,t) = \phi(r,t;\sigma(z,t))f(z,t),$$  \hspace{1cm} (2)

where the radial component is taken as

$$\phi(r,t;\sigma(z,t)) = \exp\left[\frac{-r^2}{(2\sigma^2(z,t))}\right]/\sqrt{\pi\sigma(z,t)},$$  \hspace{1cm} (3)

with a spatially and temporally variable width characterized by $\sigma(z,t)$. It should be noted here that the radial profiles of high-density cigar-shaped condensates (as the one in Ref. [12]) are closer to the Thomas-Fermi regime than to the Gaussian one. While this introduces an element of approximation to the calculation below, the adjustable nature of the parameter $\sigma(z,t)$ (which is always larger than the transverse oscillator length by a spatially dependent factor that depends on the longitudinal wave function; see Ref. [17]) and the comparison that we will report below between our theory and the physical experiment renders this approximation a reasonable one for the purpose of predicting the wavelength of the resulting pattern. We note in passing that it appears to be an interesting open problem to perform a derivation similar to that of Ref. [17], under the assumption of a transverse Thomas-Fermi wave function profile.

Employing the standard variational recipe of Ref. [17], and neglecting the longitudinal potential $U(z)$, one obtains the following effective PDE, the so-called non-polynomial Schrödinger equation (NPSE), describing the longitudinal wave function:

$$ih \frac{\partial f}{\partial t} = \left[ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + h\omega_r + \frac{1}{2}(3\epsilon_0Nf^2) \right] f.$$  \hspace{1cm} (4)

In accordance with the experimental setup of Ref. [12], the transverse frequency is modulated by

$$\omega_t(t) = \omega_{r,0}[1 + \epsilon \sin(\omega t)],$$  \hspace{1cm} (5)

where $\omega_{r,0}$ is the reference transverse frequency, and $\epsilon$ and $\omega$ are, respectively, the amplitude and frequency of the modulation. Then, the spatially homogeneous solution is given by

$$f_0(t) = A \exp\left[-ic\left(t - \epsilon \frac{\cos(\omega t)}{\omega}\right)\right],$$  \hspace{1cm} (6)

where

$$c = \frac{\omega_{r,0}}{\sqrt{1 + 2a_sNA^2}}(1 + 3a_sNA^2),$$  \hspace{1cm} (7)

and $A$ is a positive constant. The numerical value of $A$ is computed from the normalization $\int |\phi(r,z,t)|^2dr dt = 1$. Computing the integral one has that $A = \sqrt{1/2L}$, where we assume that the condensate extends between $-L$ and $L$; i.e., we are assuming here that the condensate is in a box rather than in a very weak magnetic trap in the longitudinal direction. The validity of this assumption for the present phenomenology is verified both a priori (due to the very weak nature of the longitudinal confinement in comparison to the much stronger transverse confinement and its modulation) and a posteriori (from the comparison with the experimental results).

Faraday patterns appear in this context due to a modulational instability along the (longitudinal) $z$ axis. To examine the modulational stability of uniform patterns in the $z$ direction, we use the ansatz
Inserting this ansatz into Eq. (4) and linearizing the ensuing equations yields a Mathieu-type equation for the perturbation as follows:

\[
f(t) = f_0(t)[1 + [u(t) + iv(t)]\cos(kz)].
\]  

(8)

where \(a(k, \omega)\) and \(b(k, \omega)\) are given by Eqs. (10) and (11) and \(\tau = \omega t/2\).

As is commonly known, the Mathieu equation exhibits an intricate stability chart comprising tongues of both stable and unstable solutions [18]. Due to the periodic potential a generic solution takes the form \(u(\tau) = e^{\mu \tau}g(\tau)\), where \(g(\tau)\) has the same periodicity as \(\sin(2\tau)\) (according to the Floquet-Bloch theorem), and \(\mu\) is a complex exponent. To leading order in \(a\) and \(b\) stemming from analytical means. There is a class of partly forgotten approximate formulas for \(\mu\) as a function of \(a\) and \(b\) deriving from celestial mechanics (see Ref. [20] for the main results). A convenient formula for our purpose is

\[
\mu = \frac{1}{\pi} \arccos \left[ \cos(\pi \sqrt{a}) + \frac{\pi b^2 \sin(\pi \sqrt{a})}{16 \sqrt{a(a-1)}} + O(b^4) \right],
\]

(13)

which describes accurately the first lobe of \(\mu_2\) for small values of \(b\). In order for \(\mu\) to have an imaginary part the argument of arccos must be larger than 1 (in absolute value). Naturally, identifying the most unstable mode amounts to finding the extremum value of

\[
\cos(\pi \sqrt{a}) + \frac{\pi b^2 \sin(\pi \sqrt{a})}{16 \sqrt{a(a-1)}}.
\]

To leading order in \(b\) this corresponds to \(a=1\).

Following some simple series expansions one can easily show that the first lobe of \(\mu_2\) is approximately given by

\[
\mu_2 \approx -\frac{\sqrt{b^2 - 4(a-1)^2}}{4},
\]

which indicates an instability region which extends from \(a_1 = 1-b/2\) to \(a_2 = 1+b/2\), and therefore an instability width equal to \(b\).

Finally, numerically solving Eq. (12), one readily obtains \(2\pi/k\), which represents the spacing of adjacent maxima, herein called \(\Delta\), as a function of the driving frequency \(\omega\) of the transverse confinement \(\omega_z\). Neglecting the \(k^4\) term in Eq. (10), which is numerically small under typical experimental conditions, one has that

\[
k = \omega z_{1/2} (2a, \rho)^{-1/2} (1 + 2a, \rho)^{-3/4} (4 + 6a, \rho)^{-1/2},
\]

(14)

where \(\rho = N/2L\) is the density of the condensate.
It is important to note that the above formula for $k$ [Eq. (14)] has been obtained assuming a homogeneous condensate with constant density $\rho = \rho_0$. This approximation yields a spacing between an adjacent maxima of

$$S_0 = \frac{2\pi}{k},$$

(15)

where $k$ is computed using Eq. (14) with $\rho = \rho_0$. However, the density of the condensate in the considered system is not homogeneous. To account for this inhomogeneity, as a first-order approximation, one uses the fact that, typically for the cases under consideration, the density of the condensate varies on a space scale much larger than that of the observed patterns and thus we can extend Eq. (14) with the density being space dependent: $\rho = \rho(z)$. In fact, the density of the condensate can be approximated in this slowly varying limit by the so-called Thomas-Fermi (TF) approximation as follows:

$$\rho(z) = 3\frac{L^2 - z^2}{4L^3},$$

(16)

when $-L < z < L$ and $\rho(z) = 0$ otherwise. Therefore, it is possible to approximate the average spacing by taking a spatially averaged $\bar{k}$ for $k$ in Eq. (14) using $\rho(z)$ given by the TF approximation in Eq. (16) as follows:

$$S_1 = \frac{2\pi}{\bar{k}},$$

(17)

where

$$\bar{k} = \frac{1}{2L} \int_{-L}^{L} k(z) dz.$$

(18)

Alternatively, one could average the spacing directly by using the expression

$$S_2 = \frac{1}{2L} \int_{-L}^{L} \frac{2\pi}{k(z)} dz,$$

(19)

where, again, $k(z)$ is given by Eq. (14) with the Thomas-Fermi density $\rho(z)$ of Eq. (16).

The above expressions for the spacing $S$ provide the analytical prediction of the present study that can be readily compared quantitatively with the experimental results of Ref. [12]. This comparison can be seen in Fig. 2, where the theoretical predictions for $S_0$, $S_1$, and $S_2$ defined above are adapted to the $^87$Rb experiments of Ref. [12], with $\omega_{r,0}/(2\pi) = 160.5$ Hz, the condensate length $2L = 180$ $\mu$m, and $N = 5 \times 10^9$ atoms [21]. We observe both qualitative and quantitative agreement between the theoretical predictions and the experimental result, solidifying our expectation that Eq. (14) captures accurately the observed phenomenology of the experiment of Ref. [12]. It is worth mentioning that the averages based on $S_1$ and $S_2$ are closer to the experimental data since they account for the inhomogeneity of the cloud. Notice as well that our analytical prediction shows deviations from the experimental data at low frequencies, where the spatial periods of the Faraday waves are comparable to the length of the condensate. At larger frequencies we have good agreement between the theoretical curve and the experimental data, as the periods of the Faraday waves are substantially smaller than the spatial extent of the cloud. The main sources of slight disparity between the theoretical predictions and the experimental results can be traced to the transverse Gaussian (as opposed to the Thomas-Fermi) profile and the fact that the analysis cannot directly incorporate the weak longitudinal trapping potential (see the discussion above). These will be further clarified below, through a comparison with the direct numerical simulation results of both the NPSE as well as the full 3D GP equation.

**III. NUMERICAL RESULTS**

**A. One-dimensional numerics on the NPSE**

Having completed the linear stability analysis, let us now turn to full numerical simulation to investigate the instability onset and the emergence of the relevant Faraday patterns. We have simulated the NPSE (4) for the experimental conditions described in Ref. [12]. Specifically, in Fig. 3, we show the formation of the Faraday pattern for a condensed cloud of $N = 5 \times 10^9$ $^87$Rb atoms contained in a magnetic trap with frequencies $\{\omega_{r,0}/(2\pi), \omega_{r}/(2\pi)\} = \{160.5, 7\}$ Hz, where the radial trap frequency has a modulation of 20% ($\epsilon = 0.2$, which is within the typical range of experimentally used modulations) and a frequency $\omega/(2\pi) = 321$ Hz corresponding to...
the resonant oscillation frequency of the radial breathing mode (i.e., \( \omega = 2 \omega_{r,0} \)) [22,23]. As it can be observed from the figure, the Faraday pattern grows exponentially until it is clearly visible in the density space-time evolution (bottom panel) after about 125 ms. It is reassuring that the NPSE is successful in capturing the Faraday pattern with the same clarity visible in the density space-time evolution figure, the Faraday pattern grows exponentially until it is clearly visible after some ten periods of the modulation drive.

A possible explanation for the discrepancy of the Faraday pattern growth between NPSE and the experiments may lie in the size of the initial perturbation, which will eventually seed the Faraday pattern. We have used various amplitudes for the initial perturbation after we obtained the steady state solution to Eq. (4) by imaginary time relaxation. In the results presented in this work we used an initial perturbation with an amplitude randomly chosen in an interval 0.001 times the local density. We also tried larger perturbations, up to ten times larger, and the effect is to accelerate the appearance of the patterns (results not shown here), however, we were unable to see distinguishable patterns earlier than 30–35 driving periods for the above setting. Another effect that needs to be taken into account is the amplitude of the modulation drive. While in the experiments of Ref. [12], Faraday patterns, for the resonant frequency \( \omega/(2\pi) = 321 \) Hz, quickly formed for even small drive amplitudes (less than 4%, i.e., \( \epsilon < 0.04 \)), our numerical results using the NPSE always needed a much larger drive amplitude (\( \epsilon = 0.2 \)) in the results of Fig. 3). Therefore, it is clear that the NPSE, although clearly able to capture the nature (wavelength) of the Faraday pattern, it is unable to predict the growth rate of the instability. The reason for this shortcoming stems from the fact that the transverse component of the wave function in Eq. (3) is considered to be at its ground state at all times. This is quite a strong assumption considering that the cloud presents impact oscillator-type dynamics for its radial width [12] (cf. top panel of Fig. 6 below). This will be verified when we relax the radial wave function profile in the 3D simulations shown below.

The results presented in Fig. 3 correspond to the most pattern-forming sensitive case since the drive of the radial frequency is tuned to resonate with the natural breathing frequency of the radial mode. For other driving frequencies, the growth of the Faraday pattern is less pronounced. This is demonstrated in Fig. 4, where we use the same parameter values as in Fig. 3 but change the driving frequency to \( \omega/(2\pi) = 150 \) Hz and we doubled its amplitude (\( \epsilon = 0.4 \)). For this out-of-resonance frequency, the Faraday pattern takes longer to form and even a drive with twice the amplitude takes longer to seed the pattern (the pattern is not visible until approximately 140 ms). Nonetheless, it is interesting that this out-of-resonance case only takes about 20 periods to manifest itself.
that the nonhomogeneity of the spacings induces an inherent uncertainty in the quantification of the associated spacing for a particular Faraday pattern. The associated window of spacings contained in the Faraday pattern can also be seen from the fast Fourier transform (FFT) spectrum of the density. In the bottom panels of Fig. 5 we depict the FFT spectrum of the density for a couple of cases with their respective frequency windows (see below for further elaboration on this effect).

B. Three-dimensional numerics

In order to more accurately model the Faraday pattern formation, we used direct numerical simulations of the Gross-Pitaevskii equation (1). The numerics consist of integrating Eq. (1) in cylindrical coordinates. The choice of cylindrical symmetry instead of full 3D numerics is justified by the fact that the radial direction does not develop azimuthal instabilities as can be observed in the experiments of Ref. [12] (and it was also verified by additional test runs of the full 3D equation on a coarser grid). The main challenge in numerically integrating Eq. (1) stems from the impact oscillator-type dynamics of the radial profile that is driven at resonance. These oscillations produce two numerically challenging effects: (a) they bring most of the atoms close to $r=0$ requiring an extremely fine grid, and (b) as the cloud accelerates during the impact oscillations, the wave function oscillates, in space, very rapidly (although the density does not) again requiring a very fine grid. Therefore, although (a) could be circumvented by a grid refinement around $r=0$, challenge (b) requires a fine grid where the cloud is traveling fastest and this happens on a large portion of the domain. Thus a fine grid needs to be implemented throughout the (radial) $r$ direction, and in order to ensure the stability of the numerical scheme a very small time step is also required. For the (longitudinal) $z$ direction, it suffices to have enough points to accurately capture the Faraday pattern whose wavelength is quite manageable. In the 3D simulations shown in Figs. 6 and 7 we were able to accurately integrate Eq. (1) for about seven cycles of the drive with a grid of $2001 \times 401$ points in the $(r, z)$ plane with a finite difference scheme in space with $4th–5th$ Runge Kutta in time with a time step of 0.000 25 (in adimensionalized units). The initial condition used in the simulations (i.e., the ground state of the condensate) was obtained, as for the 1D case, by imaginary time relaxation.

Figures 6 and 7 depict the Faraday pattern arising from the $(r, z)$ Gross-Pitaevskii simulation for a cloud of $N=5 \times 10^5$ $^{87}$Rb atoms driven at resonance $[\omega/(2\pi)=321 \text{ Hz}]$ by a modulation amplitude of 20%, corresponding to the experiments of Ref. [12]. Depicted in Fig. 7 is the $y$-integrated (top) view which is what is measured in the experiments. Clearly observable are the well separated fringes of the forming Faraday pattern with an approximate spacing of about 8.5 $\mu$m. Furthermore, in contrast with the NPSE simulations, the Faraday instability develops more rapidly (see the second panel in Fig. 6) and it is clearly observable after only 6–7 periods of the drive. As mentioned above, the instability sets in much more rapidly in the full 3D system than in the...
1D NPSE reduction, as expected based on the previous discussion. It is worth mentioning that in the 3D simulations we did not introduce a perturbation to the initial condition to seed the Faraday patterns since the inherent numerical noise was capable of initiating the formation of the pattern.

To validate our full 3D numerics we have computed the average spacing using the FFT method, see below for three different cases of the driving frequency. The results are depicted by the pink squares in Fig. 2. As is evidenced from the figure, the 3D numerical simulations accurately reproduce the spacing from the experiments in Ref. 12 including the case when the external driving frequency is half of the resonant frequency see the leftmost pink square in Fig. 2.

In Ref. 12, the sudden drop in the spacing around \( \omega/(2\pi)=160 \) Hz, i.e., half of the resonant frequency was attributed to excitation of the radial breathing mode. However, our numerics suggest that this is not the case and that this is due to the fact that we are driving a subharmonic that excites the resonance.

It is important to note that the data for the experiment in Ref. 12 has a significant variability in the spacing values. This natural experimental variability has many potential sources. Here, we would like to focus on the error generated by the width of the frequency peak used to measure the spacing through the fast Fourier transform (FFT). In the experiments of Ref. 12 (see the blue diamonds in Fig. 2), the spacing of the Faraday pattern was measured by computing the FFT of the integrated density image and extracting the spatial frequency of the dominant peak. The error bars depicted in Fig. 2 for the experimental data only take into account the error bar in the pixel size of the experimental snapshots (and not the variability due to the width of the FFT peak, see below).

In Fig. 8 we present a couple of examples [for \( \omega/(2\pi)=140 \) Hz and \( \omega/(2\pi)=300 \) Hz] of the Faraday patterns ob-
served in the experiments of Ref. [12]. Also in the figure, we depict the FFT of the data (second row) as well as its y-integrated counterpart (bottom row). As it is clear from the figure, there is a dominant spatial frequency that can be measured in order to extract the associated spacing of the Faraday pattern. Nonetheless, it is important to note that the dominant peak in the FFT spectrum has a width that indicates an interval of spacings that make up the original Faraday pattern. The width of this peak gives an indication of spatial variability of the pattern spacing: the wider the spectral peak the more spatial variability there exists in the pattern spacing. We have observed the same phenomenology when using our 3D numerical data. A typical example of the FFT analysis of our 3D numerical data is presented in Fig. 9. We have checked that our numerical data is able to reproduce the behavior of the FFT analysis of the experimental data. For a better comparison with experiments we added a 2.5% random noise to the numerically computed density so as to emulate the noise in the experiment (see the bottom panel of Fig. 9). In order to extract the average spacing $S_{3D}$ from the 3D numerics we computed the position of the dominant peak in the FFT (see the pink squares in Fig. 2). As it is clear from the integrated FFT curves (see the bottom two panels in Fig. 9), the Faraday pattern contains a dominant peak with a finite width. The width of the peak indicates the presence of a range of spatial frequencies (instead of a single one) and thus we can associate an error bar to the average spacing by taking the width of the dominant peak as the variability in spatial frequencies. This variability has been incorporated in the average spacing of 3D numerical data in Fig. 2 by means of the vertical error bars for the pink squares. In the same spirit, the actual error bar in the experiment should be slightly larger to also account for variability of the spatial frequency due to the width of the computed FFT peaks from the experimental data. Within this variability that, based on Figs. 8 and 9, clearly exists both in the experimental and the numerical (3D) data, we can conclude that our theoretical results are in very good agreement with both their experimental and their numerical counterparts.

**FIG. 8.** (Color online) FFT analysis for the data from the experiments of Ref. [12]. The left and right columns correspond, respectively, to a driving frequency of $\omega/(2\pi)=140$ Hz and $\omega/(2\pi)=300$ Hz. The top row depicts the images from the experiment. The second row depicts the corresponding FFTs and the third row the FFTs integrated over the y direction.

**FIG. 9.** (Color online) FFT analysis from the 3D numerics for $\omega/(2\pi)=321$ Hz. The top two panels depict, respectively, the Faraday pattern and its FFT. The third panel depicts the FFT integrated over the y direction. The bottom panel depicts the same as the third panel but the FFT is computed after adding a 2.5% noise to the density to simulate the experimental noise in the picture (notice also the difference in the depicted windows of wave numbers). The vertical lines in the last panel represent our estimated window of possible frequencies for the Faraday pattern (see the error bars in the pink squares in Fig. 2).
IV. CONCLUSIONS

In this paper, we have revisited the topic of Faraday waves and corresponding resonances in Bose-Einstein condensates. In particular, we have focused on the experimentally relevant case where the transverse confinement is periodically modulated in time. We have used the nonpolynomial Schrödinger equation as a tool that permits one to present in an explicit form this transverse modulation in an effective longitudinal equation describing the dynamics of the condensate wave function. Then, a subsequent modulational stability analysis permitted us to examine the stability of spatially uniform states in this transversely driven, yet effectively one-dimensional setting. This analysis, leading to a Mathieu equation, combined with the well-established theory of the latter equation allowed us to identify the dominant mode of instability analysis. These computations have allowed us to validate the quality of our theoretical approximations and experimental results were also corroborated by numerical calculations both within the framework of the one-dimensional NPSE equation, as well as for the case of the fully 3D Gross-Pitaevskii equation. The similarities of the two regarding the instability length scale and the differences of the two in connection to the instability growth rate have been accordingly highlighted. These computations have allowed us to validate the quality of our theoretical approximations and give a detailed comparison between theory, numerics, and experiment.

There are numerous interesting possibilities that this experiment presents for future studies. One of them is to consider the predominantly two-dimensional case of pancake-shaped condensates, where it is known that, depending on the driving frequency, square or rhombic patterns can form; see, e.g., Ref. [10], where such patterns were predicted and numerically calculated. In that setting too, the effective wave equations of Ref. [17] may allow one to perform the modulational stability analysis and obtain a quantitative handle on the dominant unstable mode. On the other hand, modulations of all three spatial dimensions of the confining potential would be of interest in their own right. In the latter case, while reductions of the type used herein would not be relevant, the dynamics may still be analytically describable upon appropriate assumptions, such as, e.g., the quadratic phase assumption of Ref. [5], by means of coupled, nonlinear ordinary differential equations such as Eqs. (12) of Ref [5]. Understanding these settings in more quantitative detail, both analytically, numerically, and experimentally, is under current examination and will be reported in future publications.

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[21] The length of the condensate is computed assuming a Thomas-Fermi density profile both radially and longitudinally. For the setup described in Refone obtains a length of 180 microns, i.e., \( L = 90 \ \mu \text{m} \).
