Q. a)

$u = x^3 + 9x + 2$

$du = 3x^2 + 9\,dx$

$du = 3(x^2+3)dx$

$\frac{1}{3}du = x^2 + 3\,dx$

$$I = \int (x^2+3)\sin(x^3+9x+2)\,dx$$

$$= \frac{1}{3}\int \sin(u)\,du$$

$$= -\frac{1}{3}\cos(u) + C$$

$$= \boxed{-\frac{1}{3}\cos(x^3+9x+2) + C.}$$

(0, b)

$$I = \int x e^x \, dx$$

$u = x \qquad dv = e^x\,dx$

$du = dx \qquad v = e^x$

$$= x e^x - \int e^x \, dx$$

$\int u\,dv = uv - \int v\,du$

$$= \boxed{x e^x - e^x + C.}$$

(1. c)

$$I = \int \frac{6x^2+8x-6}{x^3+2x^2-3x+2}\,dx$$

$u = x^3+2x^2-3x+2$

$du = 3x^2+4x-3 \; dx$

$2\,du = 6x^2+8x-6 \; dx$

$$= 2\int \frac{1}{u}\,du$$

$$= 2\ln|u| + C$$

$$= \boxed{2\ln\left|x^3+2x^2-3x+2\right| + C.}$$

①. d)

$$I = \int x\sqrt{x^2+8}\ dx \qquad u = x^2+8$$

$$du = 2x\ dx$$

$$= \frac{1}{2}\int u^{1/2}\,du \qquad \frac{1}{2}du = x\,dx$$

$$= \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{1}{3}(x^2+8)^{3/2} + C.}$$

1. a)

$$I = \int \frac{5x+1}{x^2+x-2}\,dx$$

$u = x^2+x-2$

$du = 2x+1\ dx$

(u-substitution did not help, but it was worth a try.)

$$= \int \frac{5x+1}{(x+2)(x-1)}\,dx.$$

$$\frac{5x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$5x+1 = A(x-1) + B(x+2)$$

For $x=-2$: $5(-2)+1 = A(-2-1)$

$$-9 = -3A$$

$$\boxed{A = 3}.$$

For $x=1$: $5(1)+1 = B(1+2)$

$$6 = 3B$$

$$\boxed{B=2}$$

$$I = \int \frac{3}{x+2} + \frac{2}{x-1}\,dx$$

$$= \boxed{3\ln|x+2| + 2\ln|x-1| + C.}$$

1. b)

$$I = \int \frac{2x^3+3x^2-12x-35}{x^2-x-6}\,dx .$$

Let $P(x) = 2x^3+3x^2-12x-35$, $D(x) = x^2-x-6$.

Because degree $(P) = 3 \geq$ degree $(D) = 2$,
we must use polynomial division to write

$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$, where $Q(x)$ is
the quotient and $R(x)$ is the remainder.

$\frac{P(x)}{D(x)}$:

$$
\begin{array}{r|l}
 & 2x+5 \\
\hline
x^2-x-6 & 2x^3+3x^2-12x-35 \\
 & \underline{2x^3-2x^2-12x} \\
 & 5x^2+0x-35 \\
 & \underline{5x^2-5x-30} \\
 & 5x-5
\end{array}
$$

$Q(x) = 2x+5$

$R(x) = 5x-5$

$$I = \int 2x+5+\frac{5x-5}{x^2-x-6}\,dx$$

$$= \int 2x+5\,dx + 5\int \frac{x-1}{(x-3)(x+2)}\,dx$$

$$= x^2+5x+5\int\frac{x-1}{(x-3)(x+2)}\,dx .$$

1. b) (continued)

Let $K = 5\int \frac{x-1}{(x-3)(x+2)}\,dx$.

$$\frac{x-1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$x - 1 = A(x+2) + B(x-3)$$

For $x = 3$: $3-1 = A(3+2)$

$2 = A \cdot 5$

$\boxed{A = \frac{2}{5}}$.

For $x = -2$: $-2-1 = B(-2-3)$

$-3 = -5B$

$\boxed{B = \frac{3}{5}}$

$$K = 5\int \frac{2/5}{x-3} + \frac{3/5}{x+2}\,dx$$

$$= 5\left(\frac{2}{5}\ln|x-3| + \frac{3}{5}\ln|x+2|\right) + C_1.$$

Then

$$\boxed{I = x^2 + 5x + 2\ln|x-3| + 3\ln|x+2| + C.}$$

1.c)

$$I = \int \frac{3x^4 - 2x^2 - 5x + 7}{(x^2+1)(x^2+4)^2(x-1)^2(x+1)(8x-3)}\,dx$$

$$= \underbrace{\frac{Ax+B}{x^2+1}}_{\text{single irreducible quadratic factor}} + \underbrace{\frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{x^2+4}}_{\text{repeated irreducible quadratic factor}} + \underbrace{\frac{G}{(x-1)^2} + \frac{H}{x-1}}_{\text{repeated linear factor}} + \underbrace{\frac{J}{x+1} + \frac{K}{8x-3}}_{\text{single linear factor}} + \underbrace{C_1}_{\text{Constant of Integration}}$$

2. a) $$I = \int_3^{+\infty} \frac{2}{3x^2}\,dx$$

$$= \lim_{t \to +\infty} \int_3^t \frac{2}{3x^2}\,dx$$

$$= \lim_{t \to +\infty} \left(\frac{2}{3} \int_3^t \frac{1}{x^2}\,dx \right)$$

$$= \lim_{t \to +\infty} \left(\frac{2}{3} \left[-x^{-1} \right]_3^t \right)$$

$$= \lim_{t \to +\infty} \left(\frac{2}{3} \left(\frac{1}{3} - \frac{1}{t} \right) \right)$$

$$= \lim_{t \to +\infty} \left(\frac{2}{9} - \frac{2}{3t} \right)$$

$$= \frac{2}{9}.$$

$\boxed{I \text{ converges. } I = \frac{2}{9}.}$

2. b)

Let $I = \int_2^4 \frac{2}{x-3}\,dx$.

The integrand of I, $\frac{2}{x-3}$, is discontinuous at $x = 3$.

Let $I_1 = \int_2^3 \frac{2}{x-3}\,dx$ and $I_2 = \int_3^4 \frac{2}{x-3}\,dx$.

Both of the integrals I_1 and I_2 are Improper Integrals of Type 2.

If I_1 and I_2 are convergent, then I is convergent and $I = I_1 + I_2$.

If at least one of the integrals I_1 or I_2 is divergent, then I is divergent.

The integrand of I_1 is continuous on the interval $[2, 3)$ and discontinuous at $x = 3$.

Thus, $I_1 = \lim_{t \to 3^-} \int_2^t \frac{2}{x-3}\,dx$.

2. b) (continued)

$$I_1 = \int_2^3 \frac{2}{x-3}\,dx$$

$$= \lim_{t\to 3^-} \int_2^t \frac{2}{x-3}\,dx$$

$u = x-3$
$du = dx$

Compute limits of integration

x	$u = x-3$
2	$2-3=-1$
t	$t-3$

$$= \lim_{t\to 3^-} \int_{-1}^{t-3} \frac{2}{u}\,du$$

$$= \lim_{t\to 3^-} \Big[2\ln|u| \Big]_{-1}^{t-3}$$

$$= \lim_{t\to 3^-} \Big(2\ln|t-3| - 2\ln|-1| \Big)$$

$$= \lim_{t\to 3^-} \Big(2\ln(3-t) - 2\ln(1) \Big)$$

($t\to 3^-$ means that t approaches 3 through values less than 3.
$t<3 \Rightarrow t-3<0$
$\Rightarrow |t-3| = 3-t$.)

$$= \lim_{t\to 3^-} 2\ln(3-t)$$

$$= -\infty.$$

Because I_1 is divergent, we conclude that I is divergent.

2. c)

$$I = \int_{\pi}^{+\infty} \frac{\cos^2(x)}{x^2}\, dx .$$

$$0 \leq \frac{\cos^2(x)}{x^2} \leq \frac{1}{x^2} \quad \text{for all } x \geqslant \pi .$$

$\frac{\cos^2(x)}{x^2}$ and $\frac{1}{x^2}$ are continuous for all $x \geqslant \pi$.

Let $J = \int_{\pi}^{+\infty} \frac{1}{x^2}\, dx$. Then

$$J = \lim_{t \to +\infty} \int_{\pi}^{t} \frac{1}{x^2}\, dx$$

$$= \lim_{t \to +\infty} \left[-x^{-1} \right]_{\pi}^{t}$$

$$= \lim_{t \to +\infty} \left(\frac{1}{\pi} - \frac{1}{t} \right) = \frac{1}{\pi} .$$

$\boxed{\text{J is convergent. By the Comparison Theorem, I is also convergent.}}$

2.d)

$$I = \int_1^{+\infty} \frac{1}{3x^2 + e^{2x}}\, dx.$$

For all $x \geq 1$, $\frac{1}{3x^2+e^{2x}}$ and $\frac{1}{3x^2}$ are continuous with $0 \leq \frac{1}{3x^2+e^{2x}} \leq \frac{1}{3x^2}$.

$$\int_1^{+\infty} \frac{1}{3x^2}\, dx = \frac{1}{2}\int_1^{+\infty} \frac{2}{3x^2}\, dx$$

$$= \frac{1}{2}\int_1^{3} \frac{2}{3x^2}\, dx + \frac{1}{2}\int_3^{+\infty} \frac{2}{3x^2}\, dx$$

$$= \left[-\frac{1}{3}x^{-1}\right]_1^3 + \frac{1}{2}\cdot\frac{2}{9} \quad \text{(by 2a.)}$$

$$= \left[\frac{1}{3x}\right]_3^1 + \frac{1}{9}$$

$$= \frac{1}{3} - \frac{1}{9} + \frac{1}{9}$$

$$= \frac{1}{3}.$$

~~Because~~ Thus, $\int_1^{+\infty} \frac{1}{3x^2}\, dx$ converges. By the Comparison Theorem, I converges.

2.e)

$$I = \int_1^{+\infty} \frac{2}{x} + \frac{3}{x^2}\, dx.$$

For all $x \geq 1$, $\frac{2}{x}$ and $\frac{2}{x} + \frac{3}{x^2}$ are continuous and satisfy the compound inequality

$$0 \leq \frac{1}{x} \leq \frac{2}{x} + \frac{3}{x^2}.$$

$$\int_1^{+\infty} \frac{1}{x}\, dx = \lim_{t \to +\infty} \int_1^t x^{-1}\, dx$$

$$= \lim_{t \to +\infty} \Big[\ln|x| \Big]_1^t$$

$$= \lim_{t \to +\infty} \big(\ln|t| - \ln|1| \big)$$

$$= \lim_{t \to +\infty} \big(\ln(t) \big)$$

$$= +\infty.$$

$$\boxed{\text{By the Comparison Theorem, } I \text{ diverges.}}$$

3. a)

(i) $\frac{dy}{dx} - 2xy = 2e^{x^2}$, $y(0)=1$.

Since the O.D.E. is not separable, we try an integrating factor $\mu(x) = e^{\int -2x\,dx}$. Then $\mu(x) = e^{-x^2}$, and multiplying both sides of the O.D.E. by $\mu(x)$, yields:

$\mu(x)y'(x) - 2x\mu(x)y(x) = 2e^{x^2}\mu(x) \Rightarrow$

$\mu y' - 2x\mu y = 2e^{x^2} \cdot e^{-x^2} \Rightarrow$

$\frac{d}{dx}[\mu y] = 2 \Rightarrow$

$\mu y = \int 2\,dx \Rightarrow \mu y = 2x + C \Rightarrow$

$\cancel{y = \frac{1}{\mu}\left(\int 2\,dx + C\right) \Rightarrow}$ $y = \frac{1}{\mu}(2x+C) \Rightarrow$

$y(x) = e^{x^2} \cdot 2x + Ce^{x^2}$

$\boxed{\text{(i)} \quad y(x) = (C+2x)e^{x^2} \text{ : general solution}}$

3. a) (continued)

(ii) $y(0)=1 \Rightarrow 1=(C+2(0))\cdot e^{0}$

$$\boxed{1=C}$$

$$\boxed{y(x)=(2x+1)e^{x^2} \;:\; \text{particular solution}}$$

3. b)

(i) $y'y - x\cos(x) = 0, \quad y(0)=0$

$$\frac{dy}{dx}y = x\cos(x)$$

$$\int y\,dy = \int x\cos(x)\,dx \qquad \begin{array}{ll} u=x & dv=\cos(x)dx \\ du=dx & v=\sin(x) \end{array}$$

$$\frac{1}{2}y^2 = x\sin(x) - \int \sin(x)\,dx$$

$$\frac{1}{2}y^2 = x\sin(x) + \cos(x) + C_1 \qquad C = 2C_1$$

$$\boxed{y = \pm\sqrt{2x\sin(x) + 2\cos(x) + C} \ : \text{general solution}}$$

(ii) $y(0)=0 \Rightarrow$

$$0 = 1 + C_1 \Rightarrow$$

$$C_1 = -1 \Rightarrow$$

$$C = -2 \Rightarrow$$

$$\boxed{y(x) = \pm\sqrt{2x\sin(x) + 2\cos(x) - 2} \ : \text{particular solution}}$$

3.c)

(i) $xy' - 4y = x^5 e^x, \quad y(1) = 2$

$$y' - \frac{4}{x} y = x^4 e^x, \quad x \neq 0$$

Integrating factor $\mu(x) = e^{\int -\frac{4}{x}\,dx}$

$$= e^{-4\ln|x|}$$

$$= \left(e^{\ln|x|}\right)^{-4}$$

$$= |x|^{-4}.$$

$$\frac{d}{dx}\left[\mu(x) y(x)\right] = \mu(x) x^4 e^x \Rightarrow$$

$$\frac{d}{dx}\left[\mu(x) y(x)\right] = \frac{x^4 e^x}{|x|^4} \Rightarrow$$

$$\frac{d}{dx}\left[\mu(x) y(x)\right] = e^x \Rightarrow$$

$$\mu(x) y(x) = \int e^x\,dx \Rightarrow$$

$$|x|^{-4} y(x) = e^x + C \Rightarrow$$

$$\boxed{y(x) = x^4 e^x + Cx^4 : \text{general solution}}$$

3. c) (continued)

(ii) $y(1) = 1 \Rightarrow$

$1 = 1^4 e^1 + C \cdot 1^4 \Rightarrow$

$1 = e + C \Rightarrow$

$\boxed{C = 1 - e} \Rightarrow$

$\boxed{y(x) = x^4 e^x + (1-e)x^4 \text{ : particular solution}}$

3. d)

(i) $xy^2y' = x-3, \quad y(1)=1$

$$xy^2\frac{dy}{dx} = x-3$$

$$\int y^2\,dy = \int 1-\frac{3}{x}\;dx$$

$$\frac{1}{3}y^3 = x - 3\ln|x| + C_1$$

$$y^3 = 3x - 9\ln|x| + 3C_1 \qquad C = 3C_1$$

$$\boxed{y(x) = \left(3x - 9\ln|x| + C\right)^{1/3} : \text{general solution}}$$

(ii) $y(1)=1 \Rightarrow$

$$1^3 = 3(1) - 9\ln|1| + 3C_1$$

$$1 = 3 + 3C_1$$

$$-2 = 3C_1$$

$$\boxed{C_1 = \frac{-2}{3}} \Rightarrow C = 3C_1 = 3\left(-\frac{2}{3}\right) = -2$$

$$\boxed{y(x) = \left(3x - 9\ln|x| - 2\right)^{1/3} : \text{particular solution}}$$

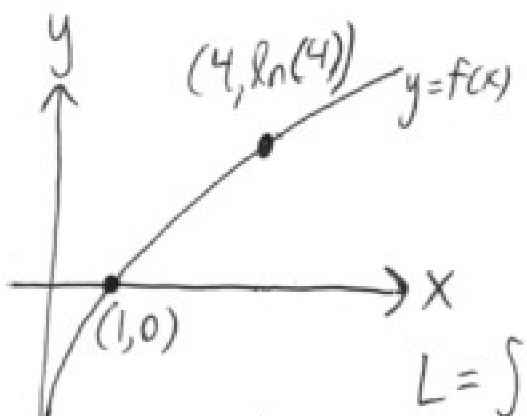
4.

Draw a graph of the curve and label the initial and final points <u>BEFORE</u> writing the integrals.

$y = f(x) = \ln(x)$, $x \in [1, 4]$.

$\ln(1) = 0$

$e < 4 < e^2 \Rightarrow \ln(e) = 1 < \ln(4) < \ln(e^2) = 2$



initial point: $(1, 0)$

final point: $(4, \ln(4))$

$y = \ln(x) \Rightarrow \frac{dy}{dx} = x^{-1}$,

$x = f^{-1}(y) = e^y$.

$e^y = x \Rightarrow \frac{dx}{dy} = e^y$.

$L = \int ds$, $ds = \sqrt{dx^2 + dy^2}$

x-integral: $L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

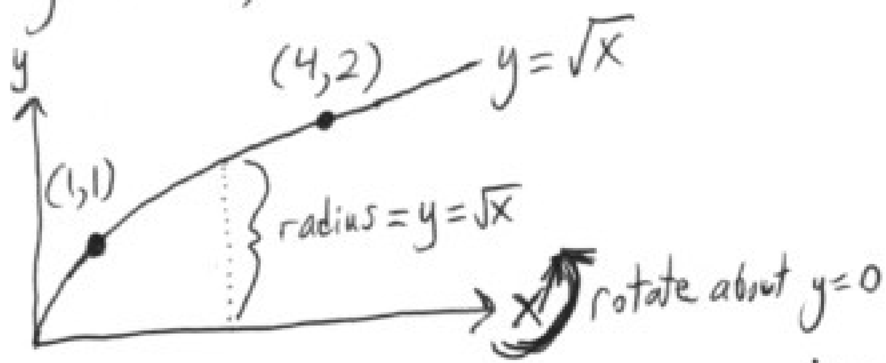
$= \boxed{\int_1^4 \sqrt{1 + x^{-2}}\, dx}$

y-integral: $L = \int_0^{\ln(4)} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

$= \boxed{\int_0^{\ln(4)} \sqrt{1 + e^{2y}}\, dy}$

5. Draw the graph <u>before</u> writing the integrals.

$y=\sqrt{x}$, $x\in[1,4]$, rotated about the x-axis ($y=0$).



initial point: (1,1)

final point: (4,2)

$ds=\sqrt{dx^2+dy^2}$.

$\frac{dy}{dx}=\frac{1}{2}x^{-1/2}$.

$y=\sqrt{x}\Rightarrow x=y^2$,

$\Rightarrow \frac{dx}{dy}=2y$.

$$S=\int \text{circumference}\ ds$$

$$=\int 2\pi(\text{radius})\,ds$$

$$=\int_1^4 2\pi\cdot\sqrt{x}\cdot\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\boxed{\int_1^4 2\pi\sqrt{x}\sqrt{1+\left(\frac{1}{2\sqrt{x}}\right)^2}\,dx : x\text{-integral}}$$

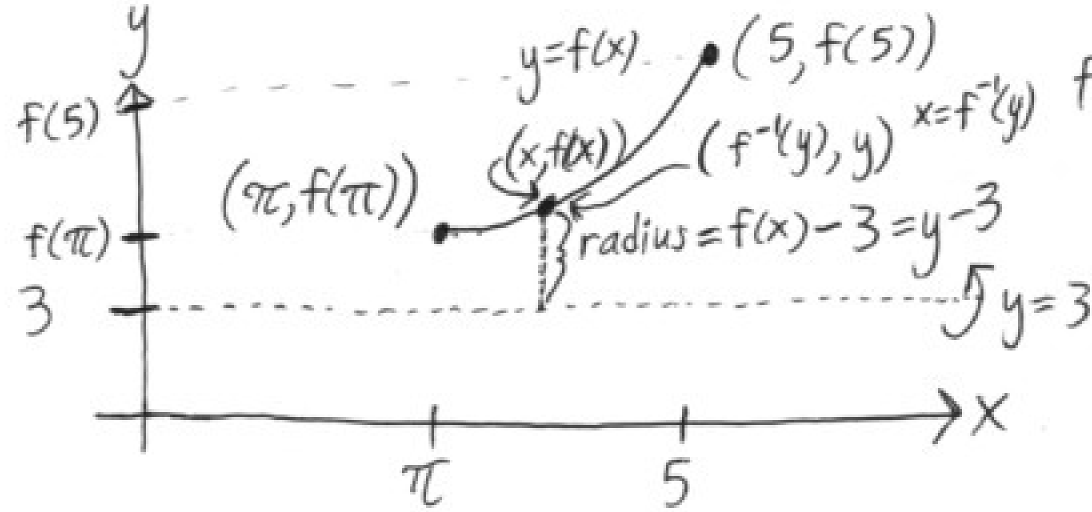
$$=\int_1^2 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=\boxed{\int_1^2 2\pi y\sqrt{1+(2y)^2}\,dy : y\text{-integral}}$$

6. Draw the graph before writing the integrals.

$y = f(x)$, $x \in [\pi, 5]$, rotated about $y = 3$.

Note: For all $x \in [\pi, 5]$, $f(x) > 3$.



initial point: $(\pi, f(\pi))$

final point: $(5, f(5))$

$ds = \sqrt{dx^2 + dy^2}$

$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy.$

$y = f(x) \Rightarrow x = f^{-1}(y).$

$\frac{dy}{dx} = f'$, $\frac{dx}{dy} = (f^{-1})'$

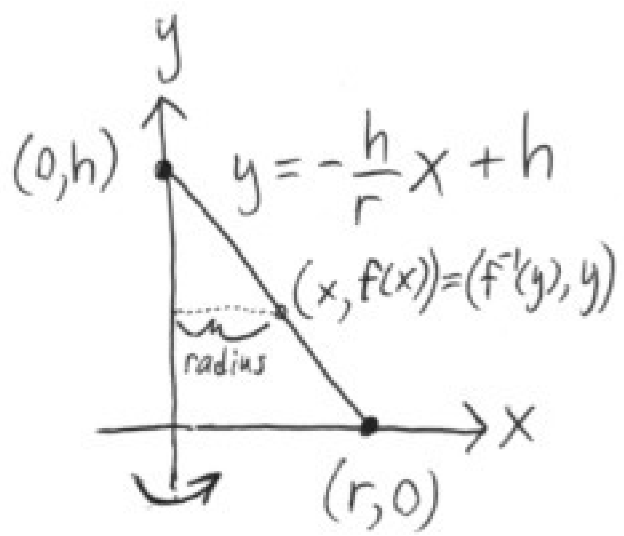
$$S = \int \text{circumference } ds$$

$$= \int 2\pi (\text{radius})\, ds$$

$$\boxed{\begin{aligned} &= \int_{\pi}^{5} 2\pi (f(x) - 3)\sqrt{1 + (f'(x))^2}\, dx \quad : x\text{-integral} \\ &= \int_{f(\pi)}^{f(5)} 2\pi (y - 3)\sqrt{1 + ((f^{-1})'(y))^2}\, dy \quad : y\text{-integral} \end{aligned}}$$

7. Consider the linear equation

$$y = -\frac{h}{r}x + h, \quad x \in [0, r].$$



Define f by $f(x) = -\frac{h}{r}x + h$. The line segment $y = f(x)$, $x \in [0, r]$, when rotated about the y-axis ($x = 0$), generates the surface of the cone with circular base of radius r and height h.

Since $y = f(x)$, $x \in [0, r]$, we may also solve for x:

$$x = f^{-1}(y) = -\frac{r}{h}(y - h) = -\frac{r}{h}y + r, \quad y \in [0, h].$$

$$\frac{dy}{dx} = f'(x) = -\frac{h}{r}, \quad \frac{dx}{dy} = (f^{-1})'(y) = -\frac{r}{h},$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy.$$

x-integral:

$$A_x = \int \text{circumference } ds$$

$$= \int 2\pi(\text{radius})\,ds$$

$$= \int_0^r 2\pi x \sqrt{1 + \left(-\frac{h}{r}\right)^2}\,dx$$

7. (continued)

$$A_x = \int_0^r 2\pi x \sqrt{1+\frac{h^2}{r^2}}\, dx$$

$$= 2\pi \sqrt{1+\frac{h^2}{r^2}} \int_0^r x\, dx$$

$$= 2\pi \frac{\sqrt{r^2+h^2}}{r} \left[\tfrac{1}{2}x^2\right]_0^r$$

$$= 2\pi \frac{\sqrt{r^2+h^2}}{r} \cdot \frac{1}{2} r^2$$

$$= \boxed{\pi r \sqrt{r^2+h^2} : \text{lateral surface area using x-integral}}$$

y-integral:

$$A_y = \int \text{circumference}\, ds$$

$$= \int 2\pi (\text{radius})\, ds$$

$$= \int_0^h 2\pi f^{-1}(y) \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_0^h 2\pi \left(-\frac{r}{h} y + r\right) \sqrt{1+\left(-\frac{r}{h}\right)^2}\, dy$$

7. (continued)

$$A_y = 2\pi\sqrt{1+\frac{r^2}{h^2}}\int_0^h -\frac{r}{h}y + r \, dy$$

$$= 2\pi\frac{\sqrt{h^2+r^2}}{h}\left[-\frac{r}{2h}y^2 + ry\right]_0^h$$

$$= 2\pi\frac{\sqrt{h^2+r^2}}{h}\left(\frac{-rh^2}{2h} + rh\right)$$

$$= 2\pi\frac{\sqrt{r^2+h^2}}{h}\cdot\left(-\frac{rh}{2} + rh\right)$$

$$= 2\pi\frac{\sqrt{r^2+h^2}}{h}\cdot\frac{rh}{2}$$

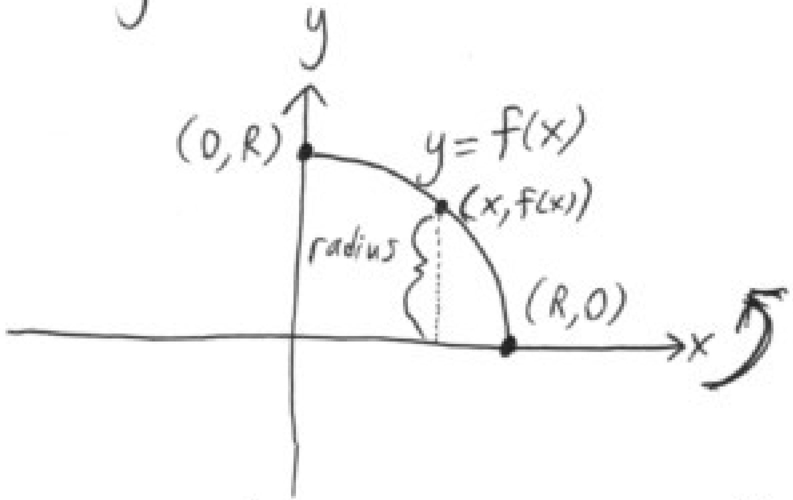
$$= \boxed{\pi r\sqrt{r^2+h^2} : \text{lateral surface area using } y\text{-integral}}$$

Regardless of the variable of integration, the lateral surface area of the cone is the desired value,

$$A_x = A_y = A = \pi r\sqrt{h^2+r^2}. \qquad \square$$

8.

Let $R>0$ be given. Define $f(x)=\sqrt{R^2-x^2}$, $x\in[0,R]$. The graph of $y=f(x)$ is the set of points in the first quadrant that are also on the circle $x^2+y^2=R^2$.



Our goal is to find the surface area of a sphere of radius R. We will first find the surface area of a hemisphere by rotating the graph of f about the x-axis. Let A_{hemi} denote the surface area of the hemisphere. Then

$$A_{hemi} = \int \text{circumference } ds$$

$$= \int 2\pi(\text{radius})\, ds.$$

We have $f'(x)=\frac{1}{2}(R^2-x^2)^{-1/2}\cdot(-2x)=-x(R^2-x^2)^{-1/2}$.

8. (continued) Then the hemisphere's area is:

$$A_{hemi} = \int_0^R 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

$$= \int_0^R 2\pi (R^2-x^2)^{1/2}\left(1+x^2(R^2-x^2)^{-1}\right)^{1/2} dx$$

$$= 2\pi \int_0^R \left((R^2-x^2)(1+x^2(R^2-x^2)^{-1})\right)^{1/2} dx$$

$$= 2\pi \int_0^R \left(R^2-x^2+x^2\right)^{1/2} dx$$

$$= 2\pi \int_0^R R\,dx$$

$$= 2\pi R\,[x]_0^R$$

$$= 2\pi R^2.$$

Since the surface area of a hemisphere is merely half the surface area of the corresponding sphere, we conclude that the surface area of a sphere of radius R is $A = 2A_{hemi} = 4\pi R^2$. $\square$

9.(a) $y(t)$ = mass of salt in kilograms at time t.

$$y'(t) = y'_{in}(t) - y'_{out}(t)$$

$$= \frac{(40\ g)}{1\ L}\left(\frac{1\ kg}{1000\ g}\right)\cdot\frac{5\ L}{1\ min.} - \left(\frac{y(t)\ kg}{100\ L}\right)\cdot\frac{5\ L}{1\ min.}$$

$$= \frac{1}{5}\ kg/min. - \frac{1}{20}\, y(t)\ kg/min.$$

$$\boxed{y'(t) = \frac{1}{5} - \frac{1}{20}y(t), \quad y(0) = 8}$$

(b)(i) $\frac{dy}{dt} = \frac{1}{5} - \frac{1}{20}y \Rightarrow$

$$20\frac{dy}{dt} = 4 - y \Rightarrow$$

$$\int \frac{20}{4-y}\,dy = \int dt$$

$$-20\int \frac{1}{y-4}\,dy = t + C_1$$

$$-20 \ln|y-4| = t + C_1$$

$$|y-4| = e^{-\frac{1}{20}(t+C_1)}$$

$$|y-4| = Ce^{-t/20}, \quad C = e^{-C_1/20}$$

9. (b)

(ii) The initial condition, $y(0)=8$, is compatible with the general solution of type $y(t)=4+Ce^{-t/20}$. We find that

$$8=4+Ce^{-0/20} \Rightarrow$$

$$8=4+C \Rightarrow$$

$$\boxed{C=4}$$

The particular solution is

$$\boxed{y(t)=4+4e^{-t/20}}$$

(iii) The amount of salt in the bucket after a very long time is 4 kg. To prove this, we ~~use~~ take the limit as $t\to+\infty$ of $y(t)$:

$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty}\left(4+4e^{-t/20}\right)$$

$$=4.$$

9. (b) (continued)

(i) If $y>4$, then $|y-4|=y-4$, and we get:

$$|y-4| = Ce^{-t/20}$$

$$y-4 = Ce^{-t/20}$$

$$\boxed{y(t) = 4 + Ce^{-t/20} \text{ : general solution for } y>4}$$

If $y<4$, then $|y-4| = 4-y$, and we get:

$$|y-4| = Ce^{-t/20}$$

$$4-y = Ce^{-t/20}$$

$$\boxed{y(t) = 4 - Ce^{-t/20} \text{ : general solution for } y<4}$$

If $y=4$, then by the O.D.E., we get:

$$\frac{dy}{dt} = \frac{1}{5} - \frac{1}{20}(4)$$

$$\frac{dy}{dt} = 0,$$

Thus $\boxed{y(t) = 4 \text{ is another solution}}$

9. (c)

$$\frac{dy}{dt} = \frac{1}{5} - \frac{1}{20}y \Rightarrow$$

$$\frac{dy}{dt} = \frac{-1}{20}(y-4).$$

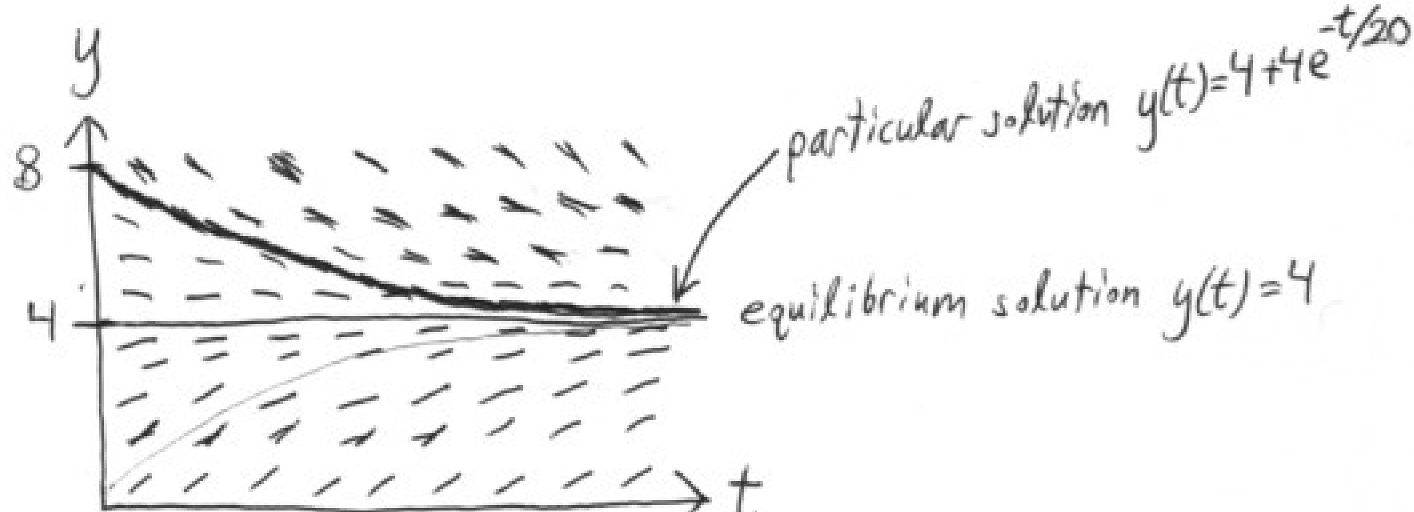
$y = 4$ is an equilibrium solution because $\frac{dy}{dt} = 0$.

$y < 4 \Rightarrow y - 4 < 0$

$\Rightarrow \frac{dy}{dt} > 0.$

$y > 4 \Rightarrow y - 4 > 0$

$\Rightarrow \frac{dy}{dt} < 0.$



10.

Let $K \in \mathbb{R}, K \geqslant 0$ be an arbitrary parameter. The family of curves defined by an equation of the form

$$x^2 + y^2 = K$$

are circles of radius $\sqrt{K}$ and center at the origin $(0,0)$.

Intuitively, we expect the family of curves orthogonal to the family of circles to be straigh lines passing through the origin. Differentiating the equation of a circle yields:

$$\frac{d}{dx}\left[x^2+y^2\right] = \frac{d}{dx}\left[K\right]$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$ ~~slope of line tangent to circle at (x,y).~~

$\frac{dy}{dx} = -\frac{x}{y}$ is the slope of the line tangent to a circle at (x,y).

The tangent lines of the family of curves orthogonal to these of the circles must have slope $\frac{dy}{dx} = \frac{-1}{-x/y} = \frac{y}{x}$.

10. (continued)

Solving the O.D.E., $\frac{dy}{dx} = \frac{y}{x}$, we find:

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx \Rightarrow$$

$$\ln|y| = \ln|x| + C_1 \Rightarrow$$

$$\ln|y| - \ln|x| = C_1 \Rightarrow$$

$$\ln\left|\frac{y}{x}\right| = C_1 \Rightarrow$$

$$\left|\frac{y}{x}\right| = e^{C_1} \Rightarrow$$

$$\frac{y}{x} = C \text{ or } \frac{y}{x} = -C, \quad C = e^{C_1} > 0.$$

Thus, the family of curves orthogonal to the family $x^2 + y^2 = K$ are lines through the origin with equations of the form $y = Cx$ or $y = -Cx$, and $C > 0$ is an arbitrary parameter.

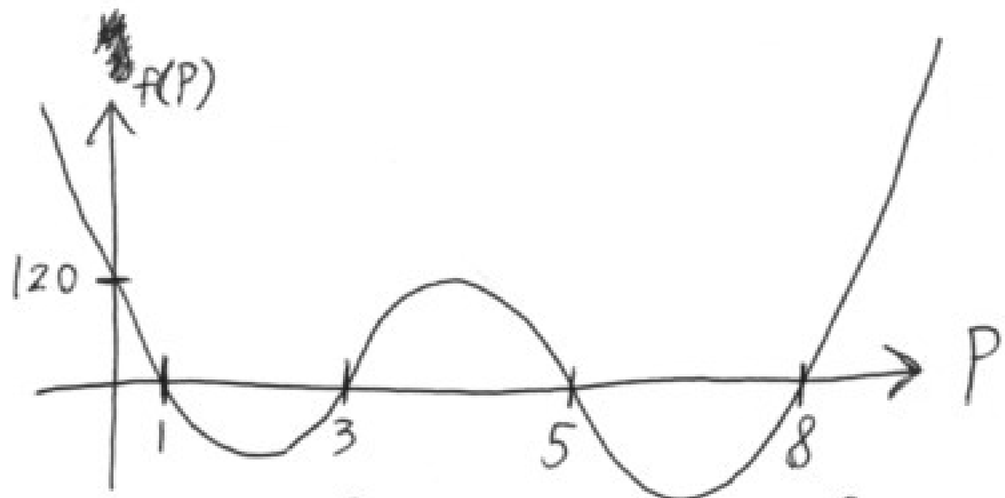
Additionally, the constant function $y = 0$ is also a solution of $\frac{dy}{dx} = \frac{y}{x}$, corresponding to the x-axis, and not included in the cases $y = Cx$ or $y = -Cx$.

11.

(a) $\frac{dP}{dt} = f(P) = (P-1)(P-3)(P-5)(P-8).$

$f(P)$ is a polynomial of degree 4 in the variable P, with positive leading coefficient.

$f(0) = (-1)(-3)(-5)(-8) = 120$: y-intercept



The roots of f are $P = 1, 3, 5, 8$.

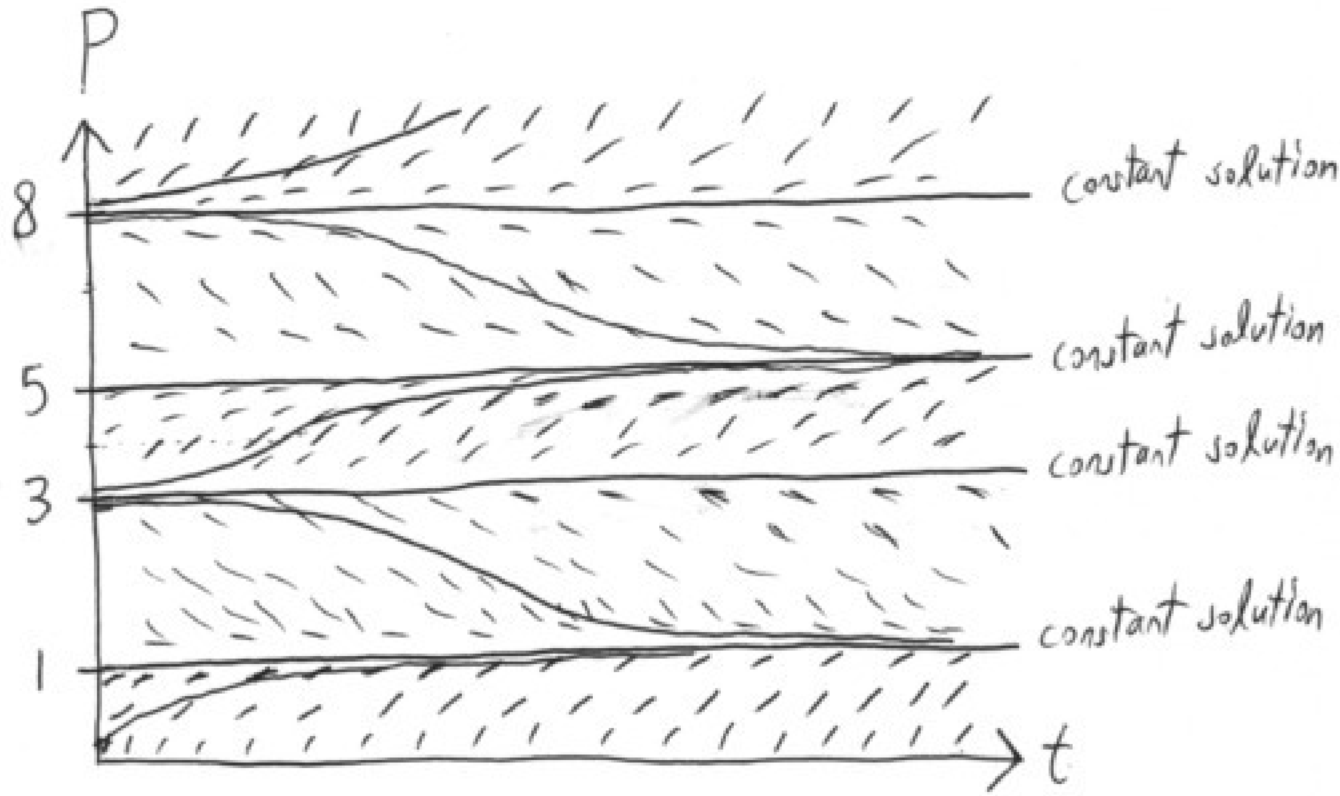
(b) (i) $P(t)$ is constant if $P = 1, 3, 5$, or 8.

(ii) $P(t)$ is increasing if $P \in (0,1)$, $P \in (3,5)$, or $P \in (8, +\infty)$.
Though $f(P) > 0$ for $P < 0$, negative values of P do not represent a population.

(iii) $P(t)$ is decreasing if $P \in (1,3)$ or $P \in (5,8)$.

11. (continued)

(c) (i), (ii)



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