REVIEW OF SEQUENCES ; SERIES

Ch. 11.1 and 11.2.

* A sequence is an (infinite) array of numbers. Many times the terms follow a pattern.

* In particular, the sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for $r < -1$ or $r > 1$. In fact:

$$\lim_{n\to\infty} \{r^n\} = \begin{cases} 0 \text{ if } |r| < 1 \\ 1 \text{ if } r = 1 \end{cases}$$

* Every bounded monotonic sequence is convergent.

* A series is a sum (infinite) of numbers. Many times its terms follow a pattern.

* Geometric Series:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$ is convergent if $|r| < 1$ and its sum is:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

If $|r| \geq 1$ the series is divergent.

* In particular: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. when $|x| < 1$

* Series test of divergence:

If $\lim_{n\to\infty} a_n$ does not exist (infinity)

or $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Ch. 11.3. The Integral Test

Let f be continuous, positive decreasing on $[1,\infty)$ and $a_n = f(n)$, then $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty} f(x)dx$ is convergent, that is:

i) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent

ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

* The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p \leq 1$.

Ch. 11.4: The Comparison Test

* Given $\sum a_n$; $\sum b_n$ with positive terms,
  - i) If $\sum b_n$ converges and $a_n \leq b_n$ for all n then $\sum a_n$ also converges
  - ii) If $\sum b_n$ diverges and $b_n \leq a_n$ for all n then $\sum a_n$ also diverges

* Limit Comparison test:

Given $\sum a_n$; $\sum b_n$ with positive terms

if $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where $c > 0$ finite,

∴ the either both series converge or diverges.

Ch 11.5: Alternating Series

* If $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots$ with $b_n > 0$

satisfies:

i) $b_{n+1} \leq b_n$ for all n

ii) $\lim_{n \to \infty} b_n = 0$

then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is convergent.

Ch. 11.6 Absolute Convergence, Ratio & Root Tests

* Def: $\sum a_n$ is absolutely convergent if $\sum |a_n|$ is convergent.

* Thm: If $\sum a_n$ is absolutely convergent then it is convergent.

Ratio Test:

i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum a_n$ is absolutely convergent.

ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$

then $\sum a_n$ is divergent.

iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, we don't know.

Root Test:

i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then $\sum a_n$ is absolutely convergent

ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or ∞, then $\sum a_n$ is divergent.

(iii) If $\lim \sqrt[n]{|a_n|} = 1$, then we don't know.

Ch. 11.8 Power Series

* Given a Power Series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there are 3 possibilities:

		Radius Conv.	Interval Conv.
i)	Series converges only for $x=a$	$\Rightarrow R=0$,	$\{a\}$
ii)	" " for all x	$\Rightarrow R=\infty$,	$(-\infty,\infty)$
iii)	" " for $\lvert x-a\rvert < R$	$\Rightarrow R>0$, finite,	$(a-R, a+R)$

* Note: The interval of convergence may include the extremes $a-R$ & $a+R$.

Ch. 11.9 Representation of Functions as Power Series

* $$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

for $|x| < 1$

Ch. 11.10 Taylor/Maclaurin Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$