1. $\left\{ \frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \dots \right\}$

numerator : $1, 2^2, 3^2, 4^2, 5^2, \dots$

denominator : $2, 3, 4, 5, 6, \dots$

alternating : $(-1)^{n+1}$

starting @ $n=1$: $a_n = \dfrac{(-1)^{n+1} n^2}{n+1}$

2.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} (-3)^{-1} \left(-\frac{3}{4}\right)^n = \sum_{n=1}^{\infty} -\frac{1}{3}\left(-\frac{3}{4}\right)^n$

$|r| = \left|-\frac{3}{4}\right| < 1$ converges to $\dfrac{\text{f.t.}}{1-r}$

$$= \frac{\frac{1}{4}}{1+3/4} = \frac{\frac{1}{4}}{7/4} = \frac{1}{7}$$

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{5^n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \frac{5^n}{3^n} = \sum_{n=1}^{\infty} \frac{1}{n^3} + \sum_{n=1}^{\infty} \left(\frac{5}{3}\right)^n$

p-series, $p=3>1$ convergent

$|r| = \frac{5}{3} > 1$ divergent geom. series

$\therefore \displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{5^n}{3^n}\right)$ is divergent.

3. $S = \displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+1)}$

$$\frac{3}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{3}{n} - \frac{3}{n+1}$$

$$3 = A(n+1) + Bn \longrightarrow A=3,\ B=-3$$

$$S_n = \sum_{i=1}^{n} \left(\frac{3}{i} - \frac{3}{i+1}\right) = \left(\frac{3}{1} - \cancel{\frac{3}{2}}\right) + \left(\cancel{\frac{3}{2}} - \cancel{\frac{3}{3}}\right) + \left(\cancel{\frac{3}{3}} - \cancel{\frac{3}{4}}\right) + \left(\cancel{\frac{3}{4}} - \frac{3}{5}\right) + \dots +$$
$$\left(\frac{3}{n-2} - \cancel{\frac{3}{n-1}}\right) + \left(\cancel{\frac{3}{n-1}} - \cancel{\frac{3}{n}}\right) + \left(\cancel{\frac{3}{n}} - \frac{3}{n+1}\right)$$

$$= 3 - \frac{3}{n+1}$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(3 - \frac{3}{n+1}\right) = 3$$

$\therefore S$ converges to 3

4. $1.\overline{8} = 1.8888\ldots = 1 + \frac{8}{10} + \frac{8}{100} + \frac{8}{1000} + \ldots$

$= 1 + \sum_{n=1}^{\infty} 8\left(\frac{1}{10}\right)^n$ geometric series with $|r| = \frac{1}{10} < 1$

$= 1 + \dfrac{\frac{8}{10}}{1 - \frac{1}{10}}$ So converges to $\frac{\text{f.t.}}{1-r}$

$= 1 + \dfrac{8/10}{9/10} = 1 + \frac{8}{9} = 1\frac{8}{9} = \frac{17}{9}$

5. (a) $Q_1 = 100$ mg

$Q_2 = 100(0.2) + 100 = 120$ mg

$Q_3 = 100(0.2)^2 + 100(0.2) + 100 = 120(0.2) + 100 = 124$ mg

(b) $Q_{n+1} = 0.2Q_n + 100$

(c) $Q_4 = 100(0.2)^3 + 100(0.2)^2 + 100(0.2) + 100$

$\vdots$

$Q_n = \sum_{i=1}^{n} 100(0.2)^{i-1}$

in the long run $= \lim_{n\to\infty} Q_n = \sum_{i=1}^{\infty} 100(0.2)^{i-1}$ convergent geometric w/ $|r| = 0.2 < 1$

$= \dfrac{100}{1 - 2/10} = \dfrac{100}{8/10} = 125$ $\frac{\text{f.t.}}{1-r}$

6. a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ positive ✓, continuous ✓, decreasing ?

decreasing? show $f'(x) < 0$ $f(x) = \frac{x}{x^2+1}$

$f'(x) = \dfrac{1(x^2+1) - 2x(x)}{(x^2+1)^2}$

$f'(x) < 0$ if $x^2 + 1 - 2x^2 < 0$

$-x^2 + 1 < 0$

$1 < x^2$

$1 < x$ ✓ decreasing on $(1, \infty)$

Now, $\int_1^\infty \frac{x}{x^2+1}\,dx$

$u = x^2+1$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$$= \int_2^\infty \frac{1}{2u}\,du = \lim_{t\to\infty}\int_2^t \frac{1}{2}\cdot\frac{1}{u}\,du = \frac{1}{2}\lim_{t\to\infty} \ln|u|\Big|_2^t$$

$$= \frac{1}{2}\lim_{t\to\infty} \ln|t| - \ln 2 = \infty$$

$\therefore$ the integral diverges.

$\therefore$ By the Integral Test, since $\int_1^\infty \frac{x}{x^2+1}\,dx$ diverges, $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ also diverges.

b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

positive ✓, continuous ✓, decreasing ?

decreasing? $f'(x)<0$ $\quad f(x) = x^2e^{-x^3}$

$$f'(x) = 2xe^{-x^3} + x^2(-3x^2e^{-x^3})$$
$$= 2xe^{-x^3} - 3x^4e^{-x^3}$$
$$= e^{-x^3}(2x-3x^4) \qquad e^{-x^3}>0$$

So, $f'(x)<0$ if $2x-3x^4<0$

if $x(2-3x^3)<0$

if $2-3x^3<0$

if $3x^3>2$

if $x^3>2/3$ ✓ decreasing for $x\geq 1$

Now, $\int_1^\infty x^2e^{-x^3}dx = \int_1^\infty \frac{x^2}{e^{x^3}}\,dx$

$u = x^3$
$du = 3x^2\,dx$
$\frac{1}{3}du = x^2dx$

$$= \int_1^\infty \frac{1}{3}\cdot e^{-u}\,du$$

$$= \lim_{t\to\infty}\int_1^t \frac{1}{3}e^{-u}\,du = \lim_{t\to\infty} -\frac{1}{3}e^{-u}\Big|_1^t = \lim_{t\to\infty}\Big[\cancel{-\frac{1}{3}e^{-t}}^{\,0} + \frac{1}{3}e^{-1}\Big] = \frac{1}{3e}$$

$\therefore$ the interval converges.

$\therefore$ By the Integral Test, since $\int_1^\infty x^2e^{-x^3}dx$ converges, $\sum_{n=1}^{\infty} n^2e^{-n^3}$ also converges.

7. a) $\sum_{n=1}^{\infty} \frac{1}{n^3+8}$ $\frac{1}{n^3+8} < \frac{1}{n^3}$ $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series, $p=3>1$.

$\therefore$ By comparison test, $\sum_{n=1}^{\infty} \frac{1}{n^3+8}$ also converges.

b) $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ $\frac{6^n}{5^n-1} > \frac{6^n}{5^n}$ $\sum_{n=1}^{\infty} \frac{6^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{6}{5}\right)^n$ is a ~~convergent~~ geometric series with $|r| = 6/5 > 1$ so it diverges.

$\therefore \sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$ diverges by comparison test.

8. a) $\sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n}$ $b_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$; $a_n = \frac{\sqrt{1+n}}{2+n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{1+n}}{2+n} \cdot \sqrt{n} = \lim_{n\to\infty} \frac{\sqrt{n+n^2}}{2+n} = \lim_{n\to\infty} \frac{\sqrt{\frac{n}{n^2}+\frac{n^2}{n^2}}}{\frac{2}{n}+\frac{n}{n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{1}{n}+1}}{\frac{2}{n}+1} = 1, \text{ finite number } > 0$$

$\therefore \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ behave similarly.

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series with $p=\frac{1}{2}<1$.

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n}$ is also divergent.

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ $b_n = \frac{\sqrt{n^4}}{n^3} = \frac{n^2}{n^3} = \frac{1}{n}$; $a_n = \frac{\sqrt{n^4+1}}{n^3+n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n^4+1}}{n^3+n} \cdot n = 1 > 0 \text{ finite number.}$$

$\therefore \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ behave similarly. $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$\therefore \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n}$ is also divergent.

③

9. a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$ $b_n = \frac{1}{3+5n}$ Check: (i) $\lim_{n\to\infty} b_n = 0$

(ii) b_n decreasing

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{3+5n} = 0$

∴ series converges by AST

(ii) $b_{n+1} = \frac{1}{3+5(n+1)}$; $b_n = \frac{1}{3+5n}$

$b_{n+1} < b_n$ since ~~the denominator of~~ $3+5(n+1) > 3+5n$

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ $b_n = \frac{n^2}{n^3+4}$ Check (i) $\lim_{n\to\infty} b_n = 0$

(ii) b_n decreasing

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+4} = \lim_{n\to\infty} \frac{n^2/n^3}{n^3/n^3 + 4/n^3} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+4/n^3} = 0$

(ii) $f(x) = \frac{x^2}{x^3+4}$ $f'(x) = \frac{2x(x^3+4) - 3x^2(x^2)}{(x^3+4)^2}$

$f'(x) < 0$ if $2x(x^3+4) - 3x^2(x^2) < 0$

if $2x^4 + 8x - 3x^4 < 0$

if $8x - x^4 < 0$

$x(8 - x^3) < 0$

if $8 - x^3 < 0$

$8 < x^3$

if $x > 2$ decreasing on $(2, \infty)$

∴ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ converges by Alternating Series Test.

10. (a) $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ $\lim_{n\to\infty} \frac{e^n}{n^2} \overset{LH}{=} \lim_{n\to\infty} \frac{e^n}{2n} \overset{LH}{=} \lim_{n\to\infty} \frac{e^n}{2} = \infty \neq 0$

∴ $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$ diverges by test for divergence

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^2+1}$ $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$ So, $\lim_{n\to\infty} (-1)^n \frac{n^2-1}{n^2+1}$ DNE.

∴ the series diverges by test for divergence.

11. (a) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

Check absolute value: $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent p-series $p = \frac{1}{2} < 1$

original series: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ $b_n = \frac{1}{\sqrt{n}}$

(i) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

(ii) $b_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = b_n$

$b_{n+1} < b_n$ since $\sqrt{n+1} > \sqrt{n}$

so b_n is decreasing

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by AST.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ is conditionally convergent.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$

Check absolute value: $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^3+1} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3+1}$ $\frac{1}{n^3+1} < \frac{1}{n^3}$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a convergent p-series with $p = 3 > 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^3+1}$ converges.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$ is absolutely convergent.

12. (a) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

Note: endpoints were not checked since the question didn't ask for it

$$L = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^n x^n} \right| = \lim_{n\to\infty} |x| \frac{n^2}{(n+1)^2} = |x| < 1$$

Radius of convergence = 1, Interval of convergence: $(-1, 1)$

12. b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n!}$

$$L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} |x| \frac{n!}{(n+1)!} = \lim_{n\to\infty} |x| \cdot \frac{1}{n+1} = 0$$

$\therefore$ radius of convergence $= \infty$; Interval : $(-\infty, \infty)$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

$$L = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right| = \lim_{n\to\infty} |x-2| \cdot \frac{n^2+1}{(n+1)^2+1} = |x-2| < 1$$

$$-1 < x-2 < 1$$

$$1 < x < 3$$

radius of convergence $= 1$

interval : $(1, 3)$

13. $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

(a) $\left(\frac{1}{1+x}\right)' = \frac{-1}{(1+x)^2}$ So, $\frac{1}{(1+x)^2} = -\left(\frac{1}{1+x}\right)'$ $\qquad \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n$

$$= -\left(\sum_{n=0}^{\infty} (-x)^n\right)'$$

$$= -\left(\sum_{n=0}^{\infty} (-1)^n x^n\right)'$$

$$= -\sum_{n=1}^{\infty} (-1)^n n x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}$$

(b) $\left(\frac{1}{(1+x)^2}\right)' = \frac{-2}{(1+x)^3}$ So, $g(x) = \frac{1}{(1+x)^3} = -\frac{1}{2}\left(\frac{1}{(1+x)^2}\right)'$

$$= -\frac{1}{2}\left(\sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1}\right)'$$

$$= -\frac{1}{2} \sum_{n=2}^{\infty} (-1)^{n+1} n(n-1) x^{n-2}$$

$$g(x) = \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1) x^{n-2}}{2}$$

c) $h(x) = \frac{x^2}{(1+x)^3}$ $\qquad h(x) = x^2 \cdot g(x) = x^2 \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1) x^{n-2}}{2}$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1) x^n}{2}$$

14. Show $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ is a solution of $f''(x) + f(x) = 0$

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$$

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1) x^{2n-2}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n-2)!}$$

$$f''(x) + f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-2}}{(2n-2)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2(n+1)-2}}{(2(n+1)-2)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} + \frac{(-1)^n x^{2n}}{(2n)!} = 0$$

15. (a) $f(x)=xe^x$, $a=0$

Taylor series:

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\frac{f^{(4)}(a)}{4!}(x-a)^4+\dots$$

$f(x)=xe^x \qquad f(0)=0$

$f'(x)=e^x+xe^x \qquad f'(0)=1$

$f''(x)=2e^x+xe^x \qquad f''(0)=2$

$f'''(x)=3e^x+xe^x \qquad f'''(0)=3$

$f^{(4)}(x)=4e^x+xe^x \qquad f^{(4)}(0)=4$

$$f(x)\approx x+x^2+\frac{x^3}{2}+\frac{x^4}{6}$$

(b) $f(x)=\sin x$, $a=\pi/6$

$f(x)=\sin x \qquad f(\frac{\pi}{6})=\frac{1}{2}$

$f'(x)=\cos x \qquad f'(\pi/6)=\sqrt{3}/2$

$f''(x)=-\sin x \qquad f''(\pi/6)=-\frac{1}{2}$

$f'''(x)=-\cos x \qquad f'''(\pi/6)=-\sqrt{3}/2$

$$f(x)\approx \frac{1}{2}+\frac{\sqrt{3}}{2}(x-\pi/6)-\frac{1}{4}(x-\frac{\pi}{6})^2-\frac{\sqrt{3}}{12}(x-\frac{\pi}{6})^3$$

~~(c) $f(x)=(1-x)^{-2}$, $a=0$~~

~~$f(x)=(1-x)^{-2} \qquad f(0)=1$~~

~~$f'(x)=-2(1-x)^{-3} \qquad f'(0)=-2$~~

~~$f''(x)=6(1-x)^{-4} \qquad f''(0)=6$~~

~~$f'''(x)=-24(1-x)^{-5} \qquad f'''(0)=-24$~~

Removed this problem from worksheet.

$$f(x)\approx 1-2x+3x^2-4x^3$$

(d) $f(x)=\ln(1+x)$, $a=0$

$f(x)=\ln(1+x) \qquad f(0)=\ln(1)=0$

$f'(x)=(1+x)^{-1} \qquad f'(0)=1$

$f''(x)=-(1+x)^{-2} \qquad f''(0)=-1$

$f'''(x)=2(1+x)^{-3} \qquad f'''(0)=2$

$f^{(4)}(x)=-6(1+x)^{-4} \qquad f^{(4)}(0)=-6$

$$f(x)\approx x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4$$

c) $f(x)=\sqrt{x}$, $a=4$

$f(x)=x^{1/2}$ $\quad f(4)=2$

$f'(x)=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$ $\quad f'(4)=\frac{1}{4}$

$f''(x)=-\frac{1}{4}x^{-3/2}=-\frac{1}{4\sqrt{x^3}}$ $\quad f''(4)=-\frac{1}{32}$

$f'''(x)=\frac{3}{8}x^{-5/2}=\frac{1}{8\sqrt{x^5}}$ $\quad f'''(4)=\frac{3}{256}$

$\cancel{f(x)\approx 4+\frac{1}{8}x+\frac{-\frac{1}{4^4}}{2!}x^2+\frac{\frac{1}{8\cdot 4^5}}{3!}x^3}$

$\cancel{f(x)\approx 4+\frac{1}{8}x-\frac{1}{512}x^2+\frac{1}{49{,}152}x^3}$

$f(x)\approx 2+\frac{1}{4}x+\frac{-\frac{1}{32}}{2!}x^2+\frac{\frac{3}{256}}{3!}x^3$

$f(x)\approx 2+\frac{1}{4}x-\frac{1}{64}x^2+\frac{1}{512}x^3$

circled x's should be replaced with (x-4)