→ *Ex ④ Find the length of the cardioid
(p. 672)

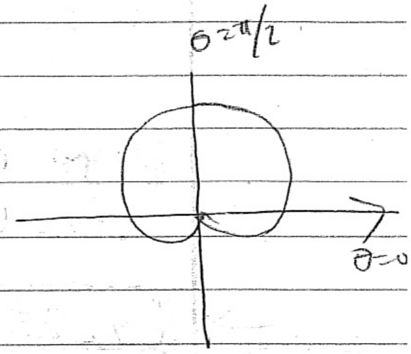
$$r = 1 + \sin\theta$$

Sol: Recall when we plotted this function, we found $0 \le \theta \le 2\pi$. Hence:

$$L = \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

$$= 2\int_{-\pi/2}^{\pi/2} \sqrt{2 + 2\sin\theta}\, d\theta = 8$$



Let's work with:

$$\sqrt{2 + 2\sin\theta}\,\frac{\sqrt{2-2\sin\theta}}{\sqrt{2-2\sin\theta}} = \frac{\sqrt{4 - 4\sin^2\theta}}{\sqrt{2-2\sin\theta}}$$

$$= \frac{2\sqrt{\cos^2\theta}}{\sqrt{2}\sqrt{1-\sin\theta}} = \sqrt{2}\,\frac{\cos\theta}{\sqrt{1-\sin\theta}}$$

Step

Hence:

$$L = 2\int_{-\pi/2}^{\pi/2} \sqrt{2+2\sin\theta}\, d\theta = \sqrt{8}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\, d\theta}{\sqrt{1-\sin\theta}}$$

$$u = 1 - \sin\theta \Rightarrow du = -\cos\theta\, d\theta$$

$$\cos\theta\, d\theta = -du$$

change the limits of integration

$$L = \sqrt{8}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\, d\theta}{\sqrt{1-\sin\theta}} = \sqrt{8}\int_{0}^{2} u^{-1/2}\, du$$

$u(-\pi/2) = 1 - \sin(-\pi/2) = 1-(-1) = 2$

$u(\pi/2) = 1 - \sin(\pi/2) = 1-1 = 0$

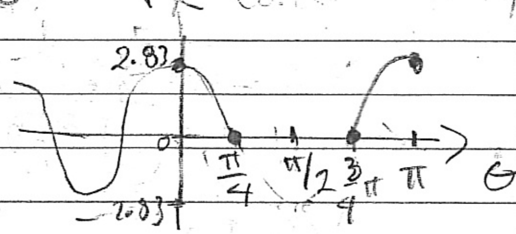
$$L = \sqrt{8}\, \frac{u^{1/2}}{1/2}\Big|_0^2 = \sqrt{32}\left[\sqrt{2}-\sqrt{0}\right] = \sqrt{64} = 8$$

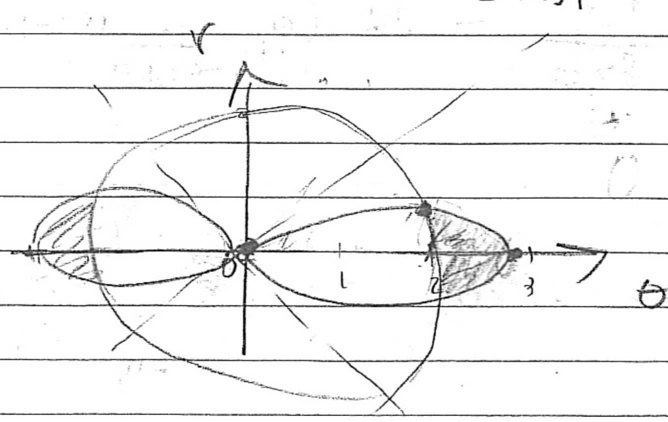
More examples of Ch. 10.4: Area and Arc Length

p. 673 → $\boxed{\text{Ex. 25}}$ Find the area of the region that lies inside of $r^2 = 8\cos 2\theta$ and outside of $r=2$.

$r = \pm\sqrt{8\cos 2\theta}$

Sol: Let's sketch.





$\cos(2\theta) = \cos(2(\pi-\theta)) = \cos 2\pi \cos 2\theta + \sin 2\pi \sin 2\theta$

$= \cos 2\theta$ ✓

Recall: $\cos(x-y) = \cos x \cos y + \sin x \sin y$

Look for points of interaction:

$r = 2$, $r^2 = 8\cos 2\theta \Rightarrow 4 = 8\cos 2\theta$

$\Rightarrow \cos 2\theta = \frac{4}{8} \Rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$

From table

$$A = 4\left[\int_0^{\pi/6} \frac{1}{2}\underbrace{(8\cos 2\theta)}_{r^2}\,d\theta - \int_0^{\pi/6} \frac{1}{2}\underbrace{(4)}_{r^2}\,d\theta\right]$$

$$= 4\left[\int_0^{\pi/6} [4\cos 2\theta - 2]\,d\theta\right]$$

$$= 4\left[\frac{4}{2}\sin 2\theta - 2\theta\right]_{\theta=0}^{\theta=\pi/6} = 8\left[\sin 2\theta - \theta\right]_{\theta=0}^{\theta=\pi/6}$$

$$= 8\left[\left(\sin \pi/3 - \frac{\pi}{6}\right) - \cancel{(\sin 0 - 0)}^{0}\right]$$

$$= 8\left[\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right] = 4\sqrt{3} - \frac{4\pi}{3} \checkmark$$

Ch. 11.10 Taylor (Maclaurin) Series:

* Ex (5) Use the def of Taylor series to find the first 3 nonzero terms of the series for $f(x)$ centered at $x=a$.

$$f(x) = xe^x \quad @ \ a=0$$

Sol: Taylor:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \overset{a=0}{=} \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4$$

Recall: $f(x) = xe^x$

$f(0) = 0e^0 = 0$

$f'(x) = xe^x + 1 \cdot e^x = (x+1)e^x \Rightarrow f'(0) = (0+1)e^0 = 1$

$f''(x) = (x+1)e^x + 1 \cdot e^x = (x+1+1)e^x = (x+2)e^x$

$\Rightarrow f''(0) = (0+2)e^0 = 2$

$f^3(x) = (x+2)e^x + 1 \cdot e^x = (x+2+1)e^x = (x+3)e^x$

$\Rightarrow f^{(3)}(0) = (0+3)e^0 = 3$

$\cancel{f^4(x) = (x+3)e^x + 1 \cdot e^x = (x+3+1)e^x = (x+4)e^x}$

$\cancel{\Rightarrow f^4(0) = (0+4)e^0 = 4}$

Hence:

$$f(x) = 0 + \frac{1}{1}x + \frac{2}{2}x^2 + \frac{3}{6}x^3$$

$$\Rightarrow f(x) = x + x^2 + \frac{1}{2}x^3 \ \checkmark\checkmark$$

Ex (37) Use Maclaurain series in table (1) to obtain the Maclaurain series for:

$f(x) = x \cos(2x)$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \rightarrow$$

$$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \quad \text{then}$$

$$f(x) = x\cos(2x) = x \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n+1}}{(2n)!} \quad ✓✓$$

Find the power series for $f'(x)$:

$$f'(x) = \frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n+1}}{(2n)!}\right]$$

$$= \sum_{n=0}^{\infty} \frac{d}{dx}\left[(-1)^n \frac{2^{2n}}{(2n)!} x^{2n+1}\right]$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{(2n)!} (2n+1) x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1) \frac{2^n (2n+1)}{(2n)!} x^{2n}$$

Ch 7: Integration Techniques

Ch 7.1: Integration by Parts

* Ex 7, p.476 Integrate:

$$\int (x^2+2x)\cos x\,dx$$

$$\begin{cases} u = x^2+2x \Rightarrow du = (2x+2)\,dx \\ dv = \cos x\,dx \Rightarrow v = \int \cos x\,dx = \sin x \end{cases}$$

$$\int u\,dv = uv - \int v\,du = (x^2+2x)\sin x - \underbrace{\int \sin x\,(2x+2)\,dx}_{\text{Parts again}}$$

$$\int \sin x\,(2x+2)\,dx$$

$$\begin{cases} u = 2x+2 \Rightarrow du = 2\,dx \\ dv = \sin x\,dx \Rightarrow v = \int \sin x\,dx = -\cos x \end{cases}$$

$$\int \sin x\,(2x+2)\,dx = -(2x+2)\cos x - \int(-\cos x)(2\,dx)$$

$$= -(2x+2)\cos x + 2\int \cos x\,dx$$

$$= -(2x+2)\cos x + 2\sin x$$

$$\int (x^2+2x)\cos x\,dx = (x^2+2x)\sin x + (2x+2)\cos x - 2\sin x + C$$

✓✓

Ch. 7.2 TRIG. INTEGRALS

* Ex⑤, p. 489

$$I = \int \sin^5(2t)\cos^2(2t)\,dt = \int \sin^4(2t)\cos^2(2t)\sin(2t)\,dt$$

$$= \int (\sin^2(2t))^2 \cos^2(2t)\sin(2t)\,dt$$

$$= \int (1-\cos^2(2t))^2 \cos^2(2t)\sin(2t)\,dt$$

$$= \int \left[1 - 2\cos^2(2t) + \cos^4(2t)\right]\cos^2(2t)\sin(2t)\,dt$$

$$\begin{cases} u = \cos(2t) & du = -2\sin(2t)\,dt \\ & \sin(2t)\,dt = -\frac{1}{2}\,du \end{cases}$$

$$I = -\frac{1}{2}\int [1 - 2u^2 + u^4]\,u^2\,du$$

$$= -\frac{1}{2}\int (u^2 - 2u^4 + u^6)\,du$$

$$= -\frac{1}{2}\left[\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right] + C$$

$$= -\frac{1}{6}\cos^3(2t) + \frac{1}{5}\cos^5(2t) - \frac{1}{14}\cos(2t) + C \checkmark$$