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More Review Exercises for the Final.

Ch. 11.2 Series (Applied Problems) Geometric

[1] * Ex (69) p. 717

A doctor prescribes a 100 mg antibiotic tablet to be taken every 8 hours. Just before each tablet is taken 20% of the drug remains in the body

(a) How much of the drug is in the body just after the second tablet is taken? After the third tablet?

Sol: (a) $Q_1 = 100$ (First tablet), $Q_2 = 100 + 0.2Q_1$ (2nd tablet) $= 100 + 0.2 \times 100$
$= 100 \cdot (1 + 0.2)$
$= 120\,mg$

$Q_3 = 100 + 0.2Q_2 = 100 + 0.2\,[100\,(1 + 0.2)]$
$= 100\,[1 + 0.2 + 0.2^2] = 100\,[1.24]$
$\boxed{Q_3 = 124\,mg}$

(b) If Q_n is the amount just after the nth tablet is taken find an equation for the Q_{n+1} in terms of Q_n:

$$Q_{n+1} = 100 + 0.2\,Q_n$$

(c) How much drug remains in the body in the long run?

$Q_2 = 100 + 0.2\,Q_1 = 100 + 0.2 \times 100 = 100\,(1 + 0.2)$
$Q_3 = 100 + 0.2\,Q_2 = 100 + 0.2\,[100\,(1 + 0.2)]$
$= 100 + 100\,(0.2 + 0.2^2) = 100\,[1 + 0.2 + 0.2^2]$

$\cdots \Rightarrow Q_n = 100 \sum_{i=0}^{n-1} (0.2)^i$

$Q_\infty = \lim_{n\to\infty} Q_n = 100 \sum_{i=0}^{\infty} (0.2)^i \overset{*}{=} 100 \sum_{j=1}^{\infty} (0.2)^{j-1} = 100 \dfrac{1}{1-r} = \dfrac{100}{1-0.2}$ ($j = i+1$, $r = 0.2$)
$= 125\,mg$

* Recall Geom. Series $\sum ar^{n-1}$

More Review Exercises for the Final

[Ch. 9.3] Differential Equations (separables)

$\boxed{2}$ * Ex (11) Find the general solution and the particular (p. 605) sol. for

$$(1)\quad \frac{dy}{dx} = xe^{y}, \quad y(0) = 0$$

Sol: $e^{-y}dy = x\,dx \Rightarrow \int e^{-y}dy = \int x\,dx$

$$\Rightarrow \quad -e^{-y} = \frac{1}{2}x^2 + c$$

[let $u = -y \Rightarrow du = -dy$, $dy = -du$ $\quad -\int e^{u}du = -e^{u} = -e^{-y}$ ✓]

$$\Rightarrow \quad e^{-y} = -\frac{1}{2}x^2 - c$$

$$\ln(e^{-y}) = \ln\left[-\frac{1}{2}x^2 - c\right]$$

$$-y = \ln\left[-\frac{1}{2}x^2 - c\right] \Rightarrow y = -\ln\left[-\frac{1}{2}x^2 - c\right]$$

$$y(0) = -\ln[-c] = 0 \Rightarrow \ln[-c] = 0 \Rightarrow -c = 1$$

$$\Rightarrow \quad c = -1$$

$$y(x) = -\ln\left[-\frac{1}{2}x^2 - (-1)\right]$$

$$= -\ln\left[1 - \frac{1}{2}x^2\right] \quad ✓✓$$

Note: You can verify by finding dy/dx and prove you get (1)

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$\boxed{3}$ Ch. 8.1 Arc length

*Ex (17) (p. 549) Find the arclength of

$$y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x) \quad \text{for} \quad 1 \le x \le 2$$

Sol: $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$$\frac{dy}{dx} = \frac{1}{2}x - \frac{1}{2x} = \frac{x^2-1}{2x}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{x^2-1}{2x}\right)^2}\, dx$$

skip

$$\sqrt{1 + \frac{(x^2-1)^2}{4x^2}} = \sqrt{\frac{4x^2 + (x^4 - 2x^2 + 1)}{4x^2}} = \sqrt{\frac{4x^2 + x^4 - 2x^2 + 1}{4x^2}}$$

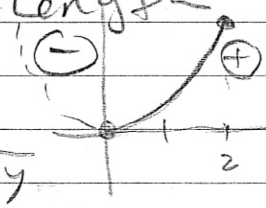
$$= \sqrt{\frac{x^4 + 2x^2 + 1}{4x^2}} = \sqrt{\frac{(x^2+1)^2}{4x^2}} = \frac{x^2+1}{2x} = \frac{1}{2}x + \frac{1}{2x}$$

$$L = \int_1^2 \left(\frac{1}{2}x + \frac{1}{2x}\right) dx = \frac{1}{4}x^2 + \frac{1}{2}\ln(x) \Big|_1^2$$

$$= \frac{1}{4} \cdot 4 + \frac{1}{2}\ln(2) - \frac{1}{4} - \frac{1}{2}\cancel{\ln(1)}^{0}$$

$$= 1 + \frac{1}{2}\ln(2) - \frac{1}{4} = \frac{3}{4} + \frac{1}{2}\ln(2) \quad \checkmark\checkmark$$

A ARC LENGTH Ch. 8.1 ARC-Length



$\boxed{4}$

$$y = \tfrac{1}{2}x^2 \quad 0 \le x \le 2$$

$$\Rightarrow x = \pm\sqrt{2y} \xrightarrow{+} +\sqrt{2y} = \sqrt{2}\sqrt{y}$$

$\boxed{X}$ $$L_x = \int ds = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$dy/dx = \frac{d}{dx}\left[\tfrac{1}{2}x^2\right] = x$$

$$\Rightarrow \boxed{L_x = \int_0^2 \sqrt{1+x^2}\,dx}$$

$\boxed{Y}$ $$L_y = \int ds = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$dx/dy = \frac{d}{dy}\left[x(y)\right] = \frac{d}{dy}\left[\sqrt{2}\sqrt{y}\right]$$

$$= \sqrt{2}\,\frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2}\sqrt{y}} = \frac{1}{\sqrt{2y}}$$

y_1: $y_1 = \tfrac{1}{2}x_1^2 = \tfrac{1}{2}0 = 0$

y_2: $y_2 = \tfrac{1}{2}x_2^2 = \tfrac{1}{2}2^2 = 2$

$$\Rightarrow \boxed{L_y = \int_0^2 \sqrt{1+\frac{1}{2y}}\,dy}$$

(B) VOLUMES Ch. 6.2

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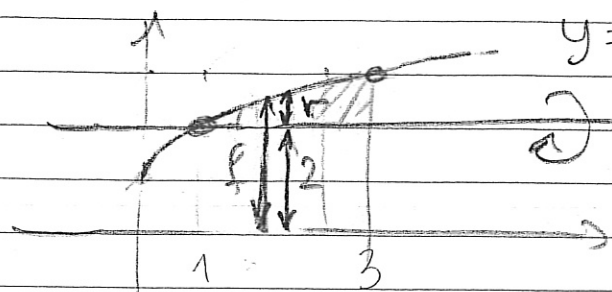


Volume generated by $y = f(x) = 1+\sqrt{x}$ rotated about $y=2$ for $1 \le x \le 3$

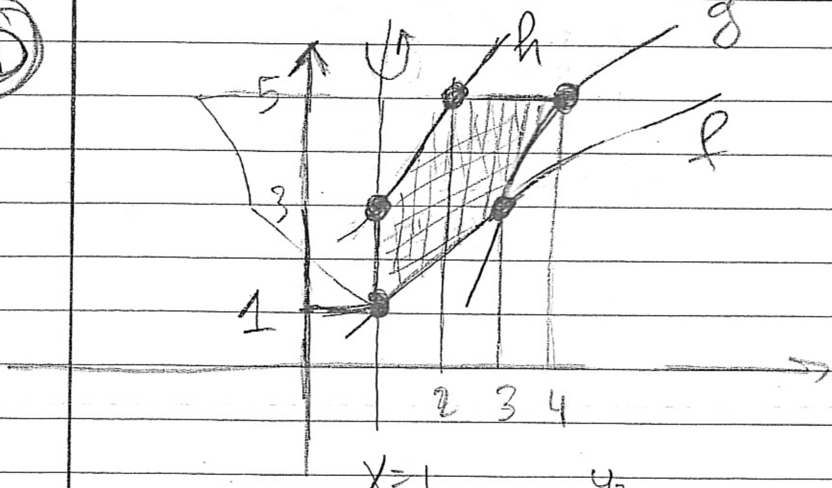
Shells: $V = \int_{x_1}^{x_2} \pi r^2 \, dx$

$r+2 = f \implies r = f(x)-2 = 1+\sqrt{x}-2$
$= \sqrt{x}-1$

DO (C) FIRST

$\therefore \boxed{V = \int_1^3 \pi (\sqrt{x}-1)^2 dx}$

(D)



Volume generated by shaded area rotating about $x=1$ for $1 \le x \le 4$

WASHERS: $V = \int_{y_1}^{y_2} \pi (r_O^2 - r_I^2) \, dy$

$V = \underbrace{\int_{y_1}^{y^*}}_{V_1} + \underbrace{\int_{y^*}^{y_2}}_{V_2}$

$y_1 = f(1) = 1$
$y^* = f(3) = g(3) = 3$
$y_2 = g(4) = 5$

V_1 | $r_I = 0$
$r_O: \quad r = x = x(y) = f^{-1}(y) - 1$

V_2 | $r_I: \quad r = x = x(y) = h^{-1}(y) - 1$
$r_O: \quad r = x = x(y) = g^{-1}(y) - 1$

$$\therefore V = \int_1^3 \pi \left(f^{-1}(y) - 1\right)^2 dy + \int_3^5 \pi \left(\left(g^{-1}(y)-1\right)^2 - \left(h^{-1}(y)-1\right)^2\right) dy$$

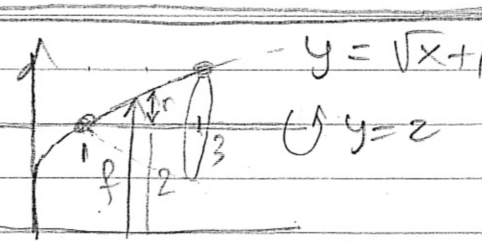
SHELLS: $\quad V = \int \text{circ.} \cdot \text{height}$
$\quad = \int 2\pi r \cdot H \, dx$

$\boxed{r}$ $\quad r = x - 1$
$\boxed{H}$ $\quad \cdot\ 0 \leq x \leq 2 \Rightarrow H = h(x) - f(x)$
$\cdot\ 2 \leq x \leq 3 \Rightarrow H = h(2) - f(x) = 5 - f(x)$
$\cdot\ 3 \leq x \leq 4 \Rightarrow H = h(2) - g(x) = 5 - g(x)$

$$\therefore V = \int_0^2 2\pi (x-1)(h(x) - f(x))\,dx$$
$$+ \int_2^3 \pi (x-1)(5 - f(x))\,dx$$
$$+ \int_3^4 \pi (x-1)(5 - g(x))\,dy$$

(C) SURFACE AREA
(same as previous)

$\boxed{6}$



$$A = \int 2\pi r \, ds = \int_{x_1}^{x_2} 2\pi r \sqrt{1 + (f'(x))^2}\,dx$$

Again: $r = f(x) - 2 = \sqrt{x} + 1 - 2 = \sqrt{x} - 1 \Rightarrow \boxed{\int_1^3 2\pi(\sqrt{x}-1)\sqrt{1 + \frac{1}{4x}}\,dx}$
$f'(x) = (\sqrt{x}+1)' = 1/2\sqrt{x} \xrightarrow{(\,)^2} (f')^2 = \frac{1}{4x}$