Sequences and Series Review

Sequences

In your own words, define what a sequence is.

____

____

A sequence can be ____________________. (infinite, finite, both)

1. Find a formula for the general term a_n, starting at $n = 0$, for the sequence $\{a_n\}_{n=0}^{\infty}$. [HINT: Transform all numbers to fractions and identify patterns for numerators and denominators.]

$$\{a_n\}_{n=0}^{\infty} = \left\{-1, \frac{4}{3}, 1, \frac{14}{27}, \frac{19}{81}, ...\right\}$$

$a_n =$

2. Find a formula for the general term a_n, starting at $n = 3$, for the sequence $\{a_n\}_{n=3}^{\infty}$. $\left[\text{HINT} : 1 = \frac{e^0}{2^0}\right]$

$$\{a_n\}_{n=3}^{\infty} = \left\{1, \frac{e}{2}, \frac{e^2}{4}, \frac{e^3}{8}, ...\right\}$$

$a_n =$

A finite sequence is simply a finite list of numbers that follow a pattern (BORING). An infinite list of numbers following a pattern (Infinite sequence) is much more interesting. One thing we do with infinite sequences is observe whether or not that list of numbers following a pattern approaches a finite number. If the limit of that sequence is a finite number, we say that the sequence CONVERGES TO THAT NUMBER, otherwise we say the sequence DIVERGES

Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} a_n = L$.

1. If $L = \infty$, then $\{a_n\}$ ____________________. (converges to L, diverges)
2. If L is a finite number, then $\{a_n\}$ ____________________. (converges to L, diverges)

Determine whether the sequence $\{a_n\}_{n=0}^{n=\infty}$ converges or diverges. If the sequence converges, what does it converge to?

1. $a_n = \dfrac{5n-1}{3^n}$

2. $a_n = \dfrac{e^{n-37}}{2^{n-520}}$

3. $a_n = \dfrac{20n^5 - 7n^4 + 1}{n - 13n^5 + 17}$

Series

A series is the sum of a ____________________.

A series can be ____________________. (infinite, finite, both)

We can denote a finite series as $S_N = \sum_{n=k}^{N} a_n$ and an infinite series as $S_\infty = \sum_{n=k}^{\infty} a_n$.

Write an expression for the sum of the first 15 even numbers. (Hint: $S_{15} = 2 + 4 + \cdots$)

$S_{15} =$ ____

Write the sum of the first 15 even numbers as a series. $\left[\text{HINT: } S_N = \sum_{k=1}^{N} a_k\right]$

$$S_{15} = \sum_{k=1}$$

Write an infinite sum for even numbers $[2 + 4 + 6 + \cdots]$ as a series with index k starting at 7. $\left[\text{HINT: } S_\infty = \sum_{k=7}^{\infty} a_k\right]$

$$S_\infty = \sum_{k=7}$$

As shown above, series notation is convenient for representing a sum of many terms that follow a pattern. One of the most interesting things that we do with series is investigate the idea of whether or not it is possible to sum up an infinite amount of numbers to never exceed a finite number. When this infinite sum of numbers approaches a finite number, we say the series

____________________ (converges, diverges), and if the infinite sum of numbers goes to positive/negative infinity,

we say the series ____________________ (converges, diverges).

The sequence $\{a_n\}_{n=4}^{\infty} = \dfrac{10n^2 + 1}{4 + 2n^2}$ converges to 5. If we sum up all the terms in that sequence we get the series $S_\infty = \sum_{k=4}^{\infty} a_k$. Since the sequence converges to 5, we can think of the series essentially adding an infinite amount of 5's in the long run. Hence,

we expect the sum of these numbers to go to ____________________ (infinity, a number, neither). Therefore the

series ____________________ (converges, diverges, neither converges nor diverges)

Fill in the blanks with <u>converges</u>, <u>diverges</u>, or <u>cannot say convergent/divergent</u> which makes the statement true.

1. If $\lim_{n\to\infty} a_n = \infty$, then $\sum_{n=k}^{\infty} a_n$ ____.
2. If $\lim_{n\to\infty} a_n = c$ (c is a non-zero number), then $\sum_{n=k}^{\infty} a_n$ ____.
3. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=k}^{\infty} a_n$ ____.

Use the results above to help you complete the following statement.

1. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n =$ _____.

Noteworthy properties of Series:

1. $\displaystyle\sum_{n=k}^{\infty} \frac{7 + 13\cos^2(2n) - 3e^n}{4^n + 1} = 7\sum_{n=k}^{\infty} \frac{1}{4^n + 1} + 13\sum_{n=k}^{\infty} \frac{\cos^2(2n)}{4^n + 1} - 3\sum_{n=k}^{\infty} \frac{e^n}{4^n + 1}$
2. $\displaystyle\sum_{n=k}^{\infty} 4^n \sin(3n) \neq \sum_{n=k}^{\infty} 4^n \sum_{n=k}^{\infty} \sin(3n)$

Three Classic Series

P-Series

$$\text{P-Series} = \sum_{n=k}^{\infty} \frac{1}{n^p}$$

1. Of the three variables, n, k, p, which one(s) are fixed numbers? ____________________

2. Of the three variables, n, k, p, which one(s) must be whole numbers? ____________________

3. The P-Series is convergent if p _____$(<, \leq, =, >, \geq)$ 1.

4. The P-Series is divergent if p _____$(<, \leq, =, >, \geq)$ 1.

Geometric Series

$$\text{Geometric Series} = \sum_{n=k}^{\infty} ar^n$$

1. Of the four variables, n, k, a, r, which one(s) are fixed numbers? ____________________

2. Of the four variables, n, k, a, r, which one(s) must be whole numbers? ____________________

3. The Geometric Series is convergent if $|r|$ _____$1(<, \leq, =, >, \geq)$

4. The Geometric Series is divergent if $|r|$ _____$1(<, \leq, =, >, \geq)$

5. Does the constant a play any role in the convergence of the Geometric Series? ____________ (Yes or No)

6. For the INFINITE Geometric Series, we know that if it converges it converges to $\dfrac{\text{"blank"}}{1 - \text{"blank"}} = \dfrac{\quad\quad}{1-\quad}$

Harmonic Series

$$\text{Harmonic Series} = \sum_{n=k}^{\infty} \frac{1}{n}$$

1. The Harmonic Series is a classic example of a series whose sequence portion tends to ____________

   (∞, zero, $-\infty$), however the series is ____________ (convergent, divergent).

The Harmonic Series illustrates how the sequence converging to zero does not imply the series will converge. Due to this non-triviality, we explore some of the tests/methods discovered by the brilliant mathematicians before us. These tests have already been proven to be true and examining why they are true is outside the scope of this course, however it is encouraged that you at least think a bit about why they make sense. Throughout the remainder of this material, try to keep in mind that we are exploring the possibility of adding an infinite amount of numbers to never exceed a finite number. If you are still having a hard time wrapping your head around this possibility, maybe the following Geometric Series will help:

$$\sum_{k=0}^{\infty} .1 \left[\frac{1}{10}\right]^k = .1 + .01 + .001 + .0001 + \cdots \qquad \text{(What number is this sum "converging" to?)}$$

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ ________________________________.

What does the Divergence Test tell us about each series? Does the test imply the series converges, diverges, or cannot say convergent/divergent.

1. $S_1 = \sum_{k=1}^{\infty} \frac{8k+7+6k^3}{12k+5}$

   S_1 ________________________________

2. $S_2 = \sum_{k=3}^{\infty} \frac{\pi^{k+2}}{e^{1-k}}$

   S_2 ________________________________

3. $S_3 = \sum_{k=0}^{\infty} \frac{\pi^k}{(k+2)!}$

   S_3 ________________________________

4. $S_4 = \sum_{k=100}^{\infty} \frac{1}{k}$

   S_4 ________________________________

5. $S_5 = \sum_{k=13}^{\infty} (\pi - e)^{-k}$

   S_5 ________________________________

The Divergence Test is one method that we can use to determine whether a series ____________________. (converges, diverges, or both)

If the sequence part of the series converges to a nonzero finite number, the Divergence Test ____________________

________________________________. (says that the series converges, says that the series diverges, says nothing about the convergence of the series)

If the sequence part of the series converges to zero, the Divergence Test ________________________________

________________________________. (says that the series converges, says that the series diverges, says nothing about the convergence of the series)

Integral Test

Fill in the blank with converges, diverges, or cannot say convergent/divergent which makes the statement true.

Let $\sum_{n=k}^{\infty} a_n$ be a series, and let $f(n) = a_n$. IF f is CONTINUOUS, EVENTUALLY STRICTLY NON-NEGATIVE, and EVENTUALLY STRICTLY DECREASING on the interval $[k, \infty)$, then

1. If we can show that $\int_k^{\infty} f(x)\, dx$ converges, then we can conclude by the Integral Test that $\sum_{n=k}^{\infty} a_n$

   ________________________.

2. If we can show that $\int_k^{\infty} f(x)\, dx$ diverges, then we can conclude by the Integral Test that $\sum_{n=k}^{\infty} a_n$

   ________________________.

"Eventually Strictly" Non-Negative and Decreasing because we can split the series into a finite series with non-positive or non-decreasing sequence part PLUS infinite series with strictly positive and decreasing sequence part.
i.e. $\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{p} a_n + \sum_{n=p}^{\infty} a_n$

Use the theorem above to fill in the blanks.
This test only works if, on the interval we are interested in "$[k, \infty)$", f is:

1. ____
2. ____
3. ____

Use the Integral Test to determine the convergence of the series $S = \sum_{k=0}^{\infty} \frac{k^3 - 2}{k^5 + 1}$

STEPS:

1. Define a continuous function for the sequence part of the series. (i.e. $a_n = f(n)$ where n is a whole number from 0 to $\infty \Rightarrow$ make a new function $f(x)$ where x is a real number between 0 and ∞)

   Let $f(x) =$

2. We clearly have that f is continuous on the interval $[0, \infty)$.
3. However, for $x \in [0, \infty)$, $f(x) < 0$ for $x \in [0, 2^{1/3})$. Rather than completely scrapping this test, we apply a little bit of ingenuity. Note that $S = \sum_{k=0}^{\infty} \frac{k^3 - 2}{k^5 + 1} = \frac{-2}{1} - \frac{1}{2} + \sum_{k=2}^{\infty} \frac{k^3 - 2}{k^5 + 1}$. If the infinite series $\overline{S} = \sum_{k=2}^{\infty} \frac{k^3 - 2}{k^5 + 1}$ converges/diverges then $S = -2 - \frac{1}{2} + \overline{S}$ must also converge/diverge. So we apply the Integral Test to this new series, with $f(x) = \frac{x^3 - 2}{x^5 + 1}$ for $x \in [2, \infty)$, which is clearly positive on the interval $[2, \infty)$.
4. Now we need to show that $f(x)$ is decreasing for $x \in [2, \infty)$ so that we meet all three criteria that enables us to use the Integral Test. Best way to show this is to take the derivative of $f(x)$ and ensure that the derivative is eventually strictly negative for $x \in [2, \infty)$.

   $f'(x) =$

   Is the derivative strictly less than zero eventually for $x \in [2, \infty)$? If so, clearly state how you know this to be true. [HINT: Plot the derivative]

   ____

   ____

5. Now that we have CLEARLY shown that all criteria have been met, we may now use the Integral Test. The integral test tells us that if $\int_2^\infty f(x)\,dx$ converges/diverges then $\overline{S}$ converges/diverges. Since $S = -2 - \frac{1}{2} + \overline{S}$, $\overline{S}$ converging/diverging implies S converges/diverges. Determine whether $\int_2^\infty f(x)\,dx$ converges or diverges. [HINT: Comparison Test for Integrals may be useful here.]

6. Therefore we have that $\int_2^\infty f(x)\,dx$ ____________________. (converges, diverges)

   Therefore we have by the Integral Test $\overline{S}$ must also ____________________. (converge, diverge)

   So finally we have that the original series, $S = \sum_{k=0}^{\infty} \frac{k^3-2}{k^5+1}$, ____________________. (converges, diverges)

Use the Integral Test to determine whether the series $S = \sum_{k=3}^{\infty} \frac{e^3 k}{\sqrt{\pi^2 k^2 - 1}}$ converges or diverges. Be sure to (1) define $f(x)$, (2) clearly state why or why not $f(x)$ satisfies the first two criteria, (3) test if the third criteria is met, [if any of the criteria are not met stop there and conclude that "Cannot apply Integral Test"] (4) show $\int_3^\infty f(x)\,dx$ converges/diverges, (5) conclude that the series S converges/diverges by the Integral Test.

Can we apply the Integral Test to the following series? If not, state why. Explain in terms of having already defined your function $f(x)$. [HINT: Are the 3 criteria met for the Integral Test?]

1. $\sum_{k=0}^{\infty} \frac{5-k}{k^3+1}$ ______________________________

2. $\sum_{k=2}^{\infty} \frac{\pi k}{k(k+2)(4-k)}$ ______________________________

3. $\sum_{k=1}^{\infty} \frac{e^k}{2^k}$ ______________________________

4. $\sum_{k=1}^{\infty} \frac{1}{k}$ ______________________________

Direct Comparison Test

Fill in the blanks with converges or diverges which makes the statement true.

Let $\sum a_n$ and $\sum b_n$ be series with $a_n, b_n \geq 0$ and $a_n \leq b_n\ \forall n$, then

1. If the larger series, $\sum b_n$, ____________________ then the smaller series, $\sum a_n$, ____________________.

2. If the smaller series, $\sum a_n$, ____________________ then the larger series, $\sum b_n$, ____________________.

If one wants to use the Direct Comparison Test for $\sum a_n$ and $\sum b_n$, a_n and b_n must be ____________________.
(continuous, positive, non-negative, convergent, divergent)

Let $\sum a_n$ and $\sum b_n$ be series with $a_n, b_n \geq 0$ and $a_n \leq b_n\ \forall n$.

1. If $\sum b_n$ converges, then the Direct Comparison Test says ______________________________.
   ($\sum a_n$ converges, $\sum a_n$ diverges, nothing about $\sum a_n$)

2. If $\sum a_n$ diverges, then the Direct Comparison Test says ______________________________.
   ($\sum b_n$ converges, $\sum b_n$ diverges, nothing about $\sum b_n$)

Use the Direct Comparison Test to determine the convergence of the series $S = \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{3\sin^2(5n)+7}{n^e+18+\cos(n\pi)}$.

STEPS:

1. What do you believe the convergence of the series to be?
   Think about the long run behavior of the sequence. Does the numerator or denominator dominate?
   If the numerator dominates, we are summing numbers that are increasing. Therefore we should expect the series to

   ____________________. (converge, diverge) If the denominator dominates, we are summing numbers that are

   decreasing in size. Therefore the series ____________________ (may, must) converge. [HINT: Think about the Harmonic Series]

   If we believe the series S converges we must find a series $\sum b_n$ that is ____________________

   (greater, less) than S that we know is ____________________. (convergent, divergent) If we believe the series

   S diverges we must find a series $\sum b_n$ that is ____________________ (greater, less) than S that we know is

   ____________________. (convergent, divergent)

   Observing the sequence part of S, we see that in the numerator we will only have value between 7 and 10. In the

   denominator, n^e is the dominating term. Thus $S \approx \sum_{n=1}^{\infty} \frac{\#}{n^e} = \# \sum_{n=1}^{\infty} \frac{1}{n^e}$. So we expect S to ____________________

   (converge, diverge). Therefore we must find a b_n that is ____________________ (greater, less) than

   $a_n = \frac{3\sin^2(5n)+7}{n^e+18+\cos(n\pi)}$. In addition, we must also know that $\sum b_n$ is ____________________. (convergent,

   divergent)

2. Since we believe S to be convergent, construct b_n by maximizing a_n.

$$0 \leq \frac{3\sin^2(5n)+7}{n^e+18+\cos(n\pi)} \leq \frac{3[1]+7}{n^e+18+[-1]} = \qquad\qquad < \qquad\qquad = b_n$$

$$0 \leq \underbrace{\qquad\qquad\qquad\qquad}_{S \text{ or } \sum b_n} \leq \underbrace{\qquad\qquad\qquad\qquad}_{S \text{ or } \sum b_n}$$

3. So we have the series $\sum b_n$ ____________________ (converges, diverges) since it is a P-Series with

   p _____ $1(\leq, >)$, and since $S = \sum_{n=3}^{\infty} \frac{3\sin^2(5n)+7}{n^e+1}$ is ____________________ (greater, less) than $\sum b_n$

   the Direct Comparison Test tells us that S is ____________________. (convergent, divergent)

Use the Direct Comparison Test to determine whether the series $S = \sum_{k=1}^{\infty} \frac{\pi^k(\sin^2(k)+7)}{3^k}$ converges or diverges. Be sure to (1) Clearly define the sequence b_k that you are comparing to the sequence part of S, (2) Show that the inequality criteria is met [i.e. both sequences are greater than or equal to zero and which of the two sequences is greater], (3) Explain how you know $\sum b_k$ is convergent/divergent, (4) Conclude that the series S converges/diverges by the Direct Comparison Test.

Key things to keep in mind when using this test:

1. We must pick a series $\sum b_n$ such that we know its convergence, therefore it is useful to select which type of series?

   ____________________________ (Harmonic Series, P-Series, Geometric Series, All three series)

2. This test can only be applied when the sequence part of the two series we are comparing are

   ____________________________ (nonzero, convergent, divergent, non-negative, positive)

3. Is it enough to know that a_n and b_n are non-negative, which of a_n and b_n is greater, and the convergence of $\sum a_n$ or $\sum b_n$ in order to apply the Direct Comparison Test to $\sum a_n$ and $\sum b_n$? Explain. [HINT: Let $a_n, b_n \geq 0$, $a_n \leq b_n$, and $\sum a_n$ be convergent. What can we say about the convergence of $\sum b_n$?]

   ____

   ____

   ____

Alternating Series

An Alternating Series is an infinite series such that the terms in the sum alternate in sign (i.e. $\# - \# + \# - \# + \cdot$). We have already established that the Harmonic Series is ____________________ (convergent, divergent), however when we add in a factor of $(-1)^n$ we get the alternating series $S = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ which is ____________________. (convergent, divergent).List the first 5 terms of each series and state whether or not the series is an Alternating Series.

Alternating Series? (Yes or No)

1. $\sum_{t=2}^{\infty} (-1)^{2t} \frac{\pi + e}{4^t} =$ ____ ____________
2. $\sum_{r=0}^{\infty} (-1)^{3r+1} \frac{\pi^2 + r}{e^r} =$ ____ ____________
3. $\sum_{k=0}^{\infty} (-1)^{k} \frac{(-5)^{4k+1}}{(-13)^k} =$ ____ ____________

The following test enables us to determine if an alternating series is convergent, granted that the series meets the criteria.

Alternating Series Test

Let $S = \sum_{n=k}^{\infty} (-1)^n a_n$ or $S = \sum_{n=k}^{\infty} (-1)^{n+1} a_n$ where $a_n \geq 0 \; \forall n$. Then if,

1. $\lim_{n \to \infty} a_n =$ _____, AND
2. a_n is a decreasing sequence

then the series S ____________________. (converges, diverges)

NOTE: To show that a_n is decreasing, set $f(n) = a_n$. If $f'(n) < 0 \; \forall n \in [k, \infty)$ then $f(n)$ is decreasing $\Rightarrow a_n$ is decreasing.

Fill in the blank with converges or diverges which makes the statement true. [Keep in mind that the Divergence Test does not specify anything about the sign of the sequence portion of the series.]

1. If $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=k}^{\infty} (-1)^n a_n$ ____.
2. Let $f(n) = a_n$. If $f'(n) > 0 \; \forall n \in [k, \infty)$ then $f(n)$ is increasing $\Rightarrow a_n$ is increasing $\Rightarrow \sum_{n=k}^{\infty} (-1)^n a_n$

   ____.

Keep in mind that this test only works if we are dealing with an Alternating Series. A cheap way of verifying is to list the first five terms in the sum and see if we get terms that alternate in sign, however, a more clever method would be to try to massage the series into the form $\sum_{n=k}^{\infty} (-1)^n a_n$ or $\sum_{n=k}^{\infty} (-1)^{n+1} a_n$ where $a_n \geq 0$. Here is an example of how to accomplish such task.

$$
\begin{aligned}
S &= \sum_{n=3}^{\infty} (-1)^{2n+1} \frac{(-5)^{4n+1}}{(-13)^{n-1}} = \sum_{n=3}^{\infty} (-1)^{2n+1} \frac{(-1)^{4n+1}(5)^{4n+1}}{(-1)^{n-1}(13)^{n-1}} = \sum_{n=3}^{\infty} (-1)^{[2n+1]+[4n+1]-[n-1]} \frac{(5)^{4n+1}}{(13)^n} \\
&= \sum_{n=3}^{\infty} (-1)^{5n+3} \frac{(5)^{4n+1}}{(13)^n} = \sum_{n=3}^{\infty} [(-1)^4]^n (-1)^n (-1)^2 (-1)^1 \frac{(5)^{4n+1}}{(13)^n} = \sum_{n=3}^{\infty} (-1)^n (-1)^1 \frac{(5)^{4n+1}}{(13)^n} = \sum_{n=3}^{\infty} (-1)^{n+1} \frac{(5)^{4n+1}}{(13)^n} \quad \checkmark
\end{aligned}
$$

Use the Alternating Series Test to prove the series $S = \sum_{n=4}^{\infty} (-1)^{3n+7} \frac{(-n)^3}{n^4+2}$ converges.

STEPS:

1. "Massage" the series into the form $S = \sum_{n=k}^{\infty} (-1)^n a_n$ or $S = \sum_{n=k}^{\infty} (-1)^{n+1} a_n$ where $a_n \geq 0 \; \forall n$ to verify that S is in fact an Alternating Series.

   $S = \sum_{n=4}^{\infty} (-1)^{3n+7} \frac{(-n)^3}{n^4+2} =$

2. Now that we have an Alternating Series in the form $\sum_{n=k}^{\infty} (-1)^n a_n$ or $\sum_{n=k}^{\infty} (-1)^{n+1} a_n$ where $a_n \geq 0$, let us verify that we meet the first criteria that $\lim_{n\to\infty} a_n = 0$.

   $\lim_{n\to\infty} a_n =$

3. The second criteria that must be met is that a_n must be a decreasing sequence. Verify that a_n is a decreasing sequence by defining a continuous function, $f(x)$ where $x \in [4, \infty)$, and verifying that the derivative is EVENTUALLY STRICTLY NEGATIVE for $x \in [4, \infty)$.

   $f(x) =$

   $f'(x) =$

   Is $f'(x) < 0 \; \forall x \in [4, \infty)$? ____________________. (Yes, No) So we have that f is an/a

   ____________________ (increasing, decreasing) ____________________ (sequence, function),

   which implies that a_n is an/a ____________________ (increasing, decreasing) ____________________.

   (sequence, function)

4. Therefore, by the ____________________ (Direct Comparison, Alternating Series, Divergence)

   Test, $S = \sum_{n=4}^{\infty} (-1)^{3n+7} \frac{(-n)^3}{n^4+2}$ ____________________. (converges, diverges)

Can the following series be "massaged" into the form $S = \sum_{n=k}^{\infty} (-1)^n a_n$ or $S = \sum_{n=k}^{\infty} (-1)^{n+1} a_n$ where $a_n \geq 0 \; \forall n$. (i.e. Is the series an Alternating Series?)

$S = \sum_{n=3}^{\infty} (-1)^{3n+1} \frac{(-5)^{4n+1}}{(-13)^{n-1}} =$

Absolutely Convergent vs. Conditionally Convergent

After the discovery of Alternating Series, two new terms were created to describe "how convergent" an Alternating Series is. These terms are Conditionally Convergent and Absolutely Convergent. We say an Alternating Series, $S = \sum_{n=k}^{\infty}(-1)^n a_n$ or $S = \sum_{n=k}^{\infty}(-1)^{n+1} a_n$ where $a_n \geq 0 \; \forall n$, is Absolutely Convergent if the sum of the absolute value of every term in the series converges. $\left(\text{i.e. } S = \sum_{n=k}^{\infty} |(-1)^n a_n| \text{ or } S = \sum_{n=k}^{\infty} |(-1)^{n+1} a_n| \text{ converges}\right)$. We say an Alternating Series, $S = \sum_{n=k}^{\infty}(-1)^n a_n$ or $S = \sum_{n=k}^{\infty}(-1)^{n+1} a_n$ where $a_n \geq 0 \; \forall n$, is Conditionally Convergent if S is not Absolutely Convergent and it passes the Alternating Series Test.

Key things to note about Absolute and Conditional Convergence:

1. ____________________ (Absolutely, Conditionally) convergent is a "better/stronger" form of

   convergence than ____________________ (Absolutely, Conditionally) convergent.

2. If a series is ____________________ (Conditionally, Absolutely) convergent then it is also

   ____________________ (Conditionally, Absolutely) convergent.

3. A convergent series consisting of strictly positive terms is inherently ____________________

   (Absolutely, Conditionally, Absolutely AND Conditionally) Convergent.

4. Due to the previous three facts, it would be wise to check for ____________________
   (Absolute, Conditional) convergence first when investigating the convergence of an Alternating Series.

Use any of the previous tests to determine whether the Alternating Series $S = \sum_{m=1}^{\infty}(-1)^{3m+2}\frac{\pi m^3}{e^{100}m^4}$ is <u>Absolutely Convergent</u>, <u>Conditionally Convergent</u>, or <u>Divergent</u>.

Limit Comparison Test

Let $\sum a_n$ and $\sum b_n$ be series with $a_n \geq 0$ and $b_n > 0 \; \forall n$. Let $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$.
Fill in the blank with both series converge, both series diverge, or test is inconclusive which makes the statement true.

Assume that we know $\sum a_n$ or $\sum b_n$ converges.

1. If $L = \infty$ or $L = 0$, then ____.
2. If L is a nonzero finite number, then ____.

Assume that we know $\sum a_n$ or $\sum b_n$ diverges.

1. If $L = \infty$ or $L = 0$, then ____.
2. If L is a nonzero finite number, then ____.

Much like the Direct Comparison Test, we must select some series such that we know it's convergence. However, the selection process for this test can be a bit more straightforward, as we will discover.

Let us investigate the convergence of the series $S = \sum\limits_{n=2}^{\infty} \frac{4\sqrt{3n^5+1} - 5n^2}{n^{3/4} + n^4 - 2}$ using the Limit Comparison Test.

STEPS:

1. Our first step is to pick a series such that we know it's convergence. One viable option is to simply guess. So let's try the Harmonic Series.

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{4\sqrt{3n^5+1} - 5n^2}{n^{3/4} + n^4 - 2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{4\sqrt{3n^5+1} - 5n^2}{n^{3/4} + n^4 - 2} \times \frac{n}{1} =$$

   Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} =$ ____________ and we know that the Harmonic Series is divergent, the Limit Comparison Test tells us ____ (nothing about S, S is convergent, S is divergent)

2. Picking the Harmonic Series did not work and we could take another random guess to test, however one trick/technique would be to look at the asymptotics (long run behavior) of the sequence. In the numerator we expect the first term to dominate out of the two terms, therefore in the numerator we have long run behavior similar to $n^{5/2}$. In the denominator we expect the second term of the three terms to dominate, therefore we have long run behavior similar to n^4. Combining the long run behavior of the numerator and denominator we get $b_n = \frac{n^{5/2}}{n^4} = \frac{n^{5/2}}{n^{8/2}} = \frac{1}{n^{3/2}}$. So let us now test the series $\sum\limits_{n=2}^{\infty} b_n$.

   $$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{4\sqrt{3n^5+1} - 5n^2}{n^{3/4} + n^4 - 2}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{4\sqrt{3n^5+1} - 5n^2}{n^{3/4} + n^4 - 2} \times \frac{n^{3/2}}{1} =$$

   Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n} =$ ____________ the Limit Comparison Test tells us ____

   (nothing about S, S and $\sum\limits_{n=2}^{\infty} \frac{1}{n^{3/2}}$ have the same convergence)

3. $\sum\limits_{n=2}^{\infty} \frac{1}{n^{3/2}}$ is a/an ____ (Geometric Series, Alternating Series, P-Series, Harmonic Series)

4. State what you know about the convergence of the series, $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$, and what is it specifically about that type of series that makes it convergent or divergent.

____

____

5. Therefore, by the Limit Comparison Test we have that the series $S = \sum_{n=2}^{\infty} \frac{4\sqrt{3n^5+1} - 5n^2}{n^{3/4} + n^4 - 2}$

____________________________ (converges, diverges).

Use the Limit Comparison Test to determine whether the series $S = \sum_{n=1}^{\infty} \frac{2^n + 13n^2 + n^{100}}{2^n n^{90}}$ converges or diverges. Be sure to (1) Clearly define the series b_n that you are "comparing/testing" with S [pick b_n by observing the asymptotics of the sequence portion of S], (2) Show that the limit of the ratio of b_n and the sequence portion of S is a finite nonzero number, (3) Explain how you know $\sum b_n$ is convergent/divergent, (4) Conclude that the series S converges/diverges by the Limit Comparison Test.

Ratio Test

Let $\sum a_n$ be a series. Let

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$$

Fill in the blank with converges, diverges, or convergence cannot be determined with this test.

1. If $L < 1$, then $\sum a_n$ ____.
2. If $L = 1$, then $\sum a_n$ ____.
3. If $L > 1$, then $\sum a_n$ ____.

Note that this test does not have any prerequisite requirements for the series being studied. Also, this test does not involve the use of another series. Therefore applying this test is fairly straightforward. The one caveat we have is

the case when L ______ $1(<,=,>)$, we have that the test is inconclusive, therefore we must

____. (attempt the Ratio Test again, try another test)

Let us investigate the convergence of the series $S = \sum\limits_{n=0}^{\infty} \frac{(-1)^n 2^{4n}}{(2n+1)!}$ using the Ratio Test.

STEPS:

1. Take the limit of the $(n+1)^{th}$ term over the n^{th} term.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}2^{4(n+1)}}{(2(n+1)+1)!}}{\frac{(-1)^n 2^{4n}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{2^{4(n+1)}}{(2(n+1)+1)!}\times\frac{(2n+1)!}{2^{4n}}\right| =$$

2. Since $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ ______ $1(<,=,>)$, the Ratio Test tells us ____________________________________.

   (nothing about S, S converges, S diverges)

Use Ratio Test to determine whether the series $S = \sum\limits_{n=1}^{\infty} \frac{(-1)^n[2\cdot 4\cdot 6\cdots(2n)]}{[2\cdot 5\cdot 8\cdots(3n-1)]}$ converges or diverges. Be sure to (1) Show all your work when taking the limit, and (2) Conclude that the series S converges/diverges or cannot be determined by the Ratio Test.