**Math 151**
**Fall 2019**
**FINAL Review**

**Name / Section:** ____________________

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1. Write an explicit integral for the volume $V$ of the solid obtained by rotating the region bounded by the given curves about the specified line.

$$y = \frac{4}{9}x^2, y = \frac{13}{9} - x^2; \text{ about the } x\text{-axis}$$

**Solution:**

$$\begin{aligned} \frac{4}{9}x^2 = \frac{13}{9} - x^2 &\iff \frac{13}{9}x^2 = \frac{13}{9} \\ &\iff x^2 = 1 \\ &\iff x = \pm 1 \end{aligned}$$

So these are the bounds for our region, and so,

$$V = \boxed{\pi \int_{-1}^{1} \left[ \left( \frac{13}{9} - x^2 \right)^2 - \left( \frac{4}{9}x^2 \right)^2 \right] \mathrm{d}x}$$

2. Use the method of cylindrical shells to write an explicit integral for the volume $V$ generated by rotating the region bounded by the given curves about the $y$-axis.

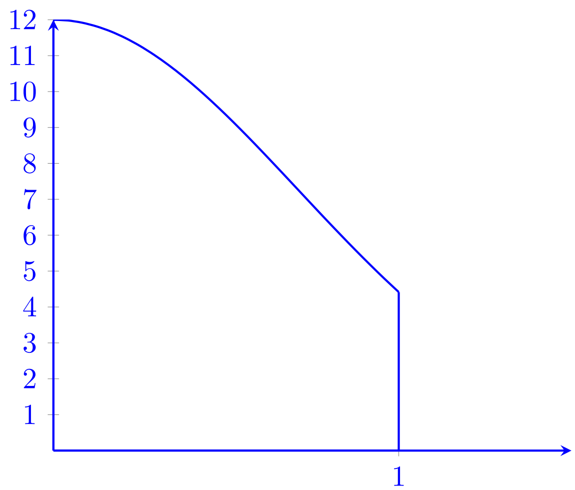
$$y = 12e^{-x^2}, y = 0, x = 0, x = 1$$

Sketch the region and a typical shell.

**Solution:**

A typical shell is a cylinder with height $12e^{-x^2}$ and a base diameter going from $-x$ to $x$, where $0 \leq x \leq 1$. All shells have a surface area of $2\pi rh$ (the top and bottom is not included) and are centered at the axis of rotation. The radius is $x$.

Anyway, we use cylindrical shells, and so




$$\begin{aligned} V = \int_0^1 2\pi rh &= \int_0^1 2\pi(x)(12e^{-x^2})\,\mathrm{d}x \\ &= \boxed{24\pi \int_0^1 xe^{-x^2}\,\mathrm{d}x} \end{aligned}$$

3. Evaluate the integral. (Use C for the constant of integration. Assume m $\neq$ 0.)

$$\int t \sinh(mt)\,\mathrm{d}t$$

**Solution:** We use integration by parts

$$u = t \qquad \mathrm{d}v = \sinh(mt)$$
$$\mathrm{d}u = \mathrm{d}t \qquad v = \frac{1}{m}\cosh(mt)$$

$$\begin{aligned}\implies \int t \sinh(mt)\,\mathrm{d}t &= \frac{t}{m}\cosh(mt) - \int \frac{1}{m}\cosh(mt)\,\mathrm{d}t \\ &= \boxed{\frac{t}{m}\cosh(mt) - \frac{1}{m^2}\sinh(mt)}\end{aligned}$$

4. Evaluate the integral. (Use C for the constant of integration.)

$$\int \sin^2(\pi x)\cos^5(\pi x)\,\mathrm{d}x$$

**Solution:**

$$\begin{aligned}\int \sin^2(\pi x)\cos^5(\pi x)\,\mathrm{d}x &= \int \sin^2(\pi x)\cos^5(\pi x)\,\mathrm{d}x \\ &= \int \sin^2(\pi x)\left(\cos^2(\pi x)\right)^2\cos(\pi x)\,\mathrm{d}x \\ &= \int \sin^2(\pi x)\left(1-\sin^2(\pi x)\right)^2\cos(\pi x)\,\mathrm{d}x \\ &= \int \sin^2(\pi x)\left(1-2\sin^2(\pi x)+\sin^4(\pi x)\right)\cos(\pi x)\,\mathrm{d}x \\ &= \int \left(\sin^2(\pi x)-2\sin^4(\pi x)+\sin^6(\pi x)\right)\cos(\pi x)\,\mathrm{d}x\end{aligned}$$

Now let $u = \sin(\pi x)$, and so $\mathrm{d}u = \pi\cos(\pi x)\,\mathrm{d}x \iff \frac{1}{\pi}\,\mathrm{d}u = \cos(\pi x)\,\mathrm{d}x$. Then we have

$$= \frac{1}{\pi}\int \left(u^2 - 2u^4 + u^6\right) \mathrm{d}u$$
$$= \frac{1}{\pi}\left(\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right) + C$$
$$= \boxed{\frac{1}{\pi}\left(\frac{1}{3}\sin^3(\pi x) - \frac{2}{5}\sin^5(\pi x) + \frac{1}{7}\sin^7(\pi x)\right) + C}$$

5. Evaluate the integral. (Remember to use absolute values where appropriate. Use C for the constant of integration.)

$$\int \frac{3x^2 - 19x + 46}{(2x+1)(x-2)^2} dx$$

**Solution:** We are going to use integration by partial fraction decomposition. We already have a factorized denominator, and in fact, we have one unique and one twice-repeated factor. So...

$$\frac{3x^2 - 19x + 46}{(2x+1)(x-2)^2} = \frac{A}{2x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$
$$\iff 3x^2 - 19x + 46 = A(x-2)^2 + B(2x+1)(x-2) + C(2x+1)$$
$$\iff 3x^2 - 19x + 46 = A(x^2 - 4x + 4) + B(2x^2 - 3x - 2) + C(2x+1)$$
$$\iff 3x^2 - 19x + 46 = (A + 2B)x^2 + (-4A + -3B + 2C)x + (4A - 2B + C)$$

So we arrive at the system of equations:

$$\begin{cases} A + 2B = 3 \\ -4A - 3B + 2C = -19 \\ 4A - 2B + C = 46 \end{cases}$$

$$\begin{cases} A = 3 - 2B \\ -4(3 - 2B) - 3B + 2C = -19 \\ 4(3 - 2B) - 2B + C = 46 \end{cases}$$

$$\begin{cases} A = 3 - 2B \\ -12 + 8B - 3B + 2(34 + 10B) = -19 \\ C = 34 + 10B \end{cases}$$

$$\begin{cases} A = 3 - 2B \\ 25B = -75 \\ C = 34 + 10B \end{cases}$$

From the above we get that $A = 9$, $B = -3$, and $C = 4$, and so we have

$$\begin{aligned}\int \frac{3x^2 - 19x + 46}{(2x+1)(x-2)^2}\,\mathrm{d}x &= \int \left(\frac{9}{2x+1} + \frac{-3}{x-2} + \frac{4}{(x-2)^2}\right)\mathrm{d}x \\ &= \boxed{\frac{9}{2}\ln|2x+1| - 3\ln|x-2| - \frac{4}{x-2} + C}\end{aligned}$$

6. Evaluate the integral. (Use C for the constant of integration.)

$$\int x^3\sqrt{16 - x^2}\,dx$$

**Solution:** This is a trig sub problem. We want to represent $\sqrt{16 - x^2}$ as a side in a right triangle. Draw the triangle on your own paper for practice.

We will call one of the non-90° angles $\theta$. Its opposite side will be $x$, its adjacent side will be $\sqrt{16 - x^2}$, and its hypotenuse will be 4. Use the Pythagorean Theorem to check that $a^2 + b^2 = c^2$. Now note:

$$\begin{aligned}\sin(\theta) &= x/4 \\ x &= 4\sin(\theta) \\ \mathrm{d}x &= 4\cos(\theta)\,\mathrm{d}\theta\end{aligned}$$

and:

$$\begin{aligned}\cos(\theta) &= \frac{\sqrt{16 - x^2}}{4} \\ \sqrt{16 - x^2} &= 4\cos(\theta)\end{aligned}$$

Substituting into the original integral gives:

$$\begin{aligned}&\int 4^5 \sin^3\theta \cos^2\theta \ \mathrm{d}\theta \\ &= \int 4^5 \sin\theta(1 - \cos^2\theta)\cos^2\theta \ \mathrm{d}\theta\end{aligned}$$

Now use a u-sub with $u = \cos(\theta)$ and $\mathrm{d}u = -\sin(\theta)\,\mathrm{d}\theta$.

$$= \int -4^5(1-u^2)u^2 \, \mathrm{d}u$$
$$= \int 4^5(u^4-u^2) \, \mathrm{d}u$$
$$= 4^5\left[\frac{u^5}{5}-\frac{u^3}{3}\right]$$
$$= 4^5\left[\frac{\cos^5\theta}{5}-\frac{\cos^3\theta}{3}\right]$$
$$= 4^5\left[\frac{(16-x^2)^{5/2}}{5*4^5}-\frac{(16-x^2)^{3/2}}{3*4^3}\right]$$
$$= \frac{(16-x^2)^{5/2}}{5}-\frac{16(16-x^2)^{3/2}}{3}$$

7. If the infinite curve $y=e^{-3x}, x \geq 0$ is rotated about the $x$-axis, find the surface area of the resulting surface.

**Solution:** This problem is really multiple problems in one, and covers many topics from Midterms 1 and 2.

$$\text{Surface Area}_{a\leq x\leq b} = 2\pi\int_a^b y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$
$$= 2\pi\int_0^\infty e^{-3x}\sqrt{1+(-3e^{-3x})^2}\,\mathrm{d}x$$
$$= 2\pi\lim_{t\to\infty}\int_0^t e^{-3x}\sqrt{1+9e^{-6x}}\,\mathrm{d}x$$

Now let $u=3e^{-3x}$, then $\mathrm{d}u=-9e^{-3x}\,\mathrm{d}x \iff -\frac{1}{9}\,\mathrm{d}u = e^{-3x}\,\mathrm{d}x$, and so after changing our bounds of integration, we have

$$= -\frac{2\pi}{9}\lim_{t\to\infty}\int_3^{3e^{-3t}}\sqrt{1+u^2}\,\mathrm{d}u$$

Now we will make the substitution $u=\tan\theta$, then $\mathrm{d}u=\sec^2(\theta)\,\mathrm{d}\theta$. So...

$$= -\frac{2\pi}{9}\lim_{t\to\infty}\int_{\arctan(3)}^{\arctan(3e^{-3t})}\sqrt{1+\tan^2(\theta)}\sec^2(\theta)\,\mathrm{d}\theta$$
$$= -\frac{2\pi}{9}\lim_{t\to\infty}\int_{\arctan(3)}^{\arctan(3e^{-3t})}\sqrt{\sec^2(\theta))}\sec^2(\theta)\,\mathrm{d}\theta$$
$$= -\frac{2\pi}{9}\lim_{t\to\infty}\int_{\arctan(3)}^{\arctan(3e^{-3t})}\sec^3(\theta)\,\mathrm{d}\theta$$

So it could be that you have derived $\int \sec^3\theta \, d\theta$ before, but either way, it is kind of long and an exercise on its own. You can find the proof at https://www.math.ubc.ca/ feldman/m121/secx.pdf.

$$= -\frac{2\pi}{9} \lim_{t\to\infty} \frac{1}{2} \left(\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right) \Big|_{\arctan(3)}^{\arctan(3e^{-3t})}$$

At this point, notice that $\tan(\arctan(x)) = x$ for any x, and $\sec(\arctan(x)) = \sqrt{1+x^2}$ for any $x$ (draw out a triangle and use $\tan(\theta) = x = x/1 = \textit{opposite}/\textit{adjacent}$). So then we have

$$\begin{aligned}
&= -\frac{\pi}{9} \lim_{t\to\infty} \left[\left((\sqrt{1+9e^{-6t}})(3e^{-3t}) + \ln|\sqrt{1+9e^{-6t}} + 3e^{-3t}\right) - \left((\sqrt{1+3^2})(3) + \ln|\sqrt{1+3^2}+3|\right)\right] \\
&= -\frac{\pi}{9} \left[\left((\sqrt{1})(0) + \ln|\sqrt{1}+0|\right) - \left((\sqrt{1+3^2})(3) + \ln|\sqrt{1+3^2}+3|\right)\right] \\
&= -\frac{\pi}{9} \left[0 - \left((3\sqrt{10}) + \ln(\sqrt{10}+3)\right)\right] \\
&= \boxed{\frac{\pi}{9}\left((3\sqrt{10}) + \ln(\sqrt{10}+3)\right)}
\end{aligned}$$

8. The air in a room with volume 180 m$^3$ contains 0.15% carbon dioxide initially. Fresher air with only 0.05% carbon dioxide flows into the room at a rate of 2 m$^3$/min and the mixed air flows out at the same rate.

   Find the percentage of carbon dioxide in the room as a function of time t (in minutes).

   What happens with the percentage of carbon dioxide in the room in the long run?

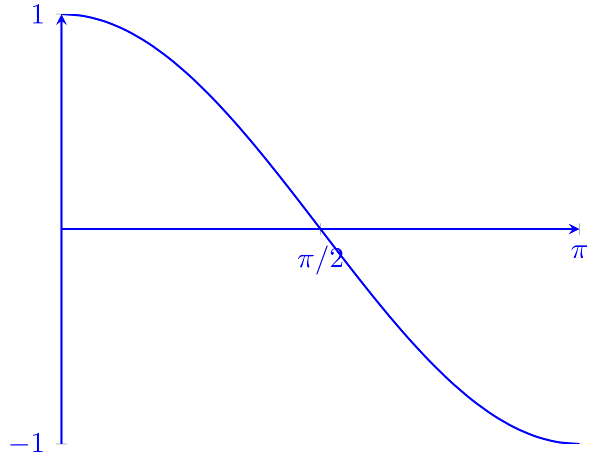
   **Solution:** NOT READY YET

9. Write TWO explicit integrals, one in $x$ and one in $y$, for the length of the curve $y = \cos(x)$ for $0 \le x \le \pi$. DRAW A SKETCH!

   **Solution:**

$$\begin{aligned}
L &= \int_{x_{\min}}^{x_{\max}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{\pi} \sqrt{1 + \left(\frac{d}{dx}\cos(x)\right)^2}\, dx \\
&= \int_0^{\pi} \sqrt{1 + (-\sin(x))^2}\, dx \\
&= \boxed{\int_0^{\pi} \sqrt{1 + \sin^2(x)}\, dx}
\end{aligned}$$

$$\begin{aligned}
L &= \int_{y_{\min}}^{y_{\max}} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\
&= \int_{-1}^{1} \sqrt{1 + \left(\frac{d}{dy}\arccos(y)\right)^2}\, dy \\
&= \int_{-1}^{1} \sqrt{1 + \left(\frac{-1}{\sqrt{1-y^2}}\right)^2}\, dy \\
&= \boxed{\int_{-1}^{1} \sqrt{1 + \frac{1}{1-y^2}}\, dy}
\end{aligned}$$




10. Solve the initial value problem:

$$xy' = y + 3x^2 \sin x,\ \ y(\pi) = 0$$

**Solution:**

This problem is not separable, and requires an integrating factor. Note that we have to get the problem into the correct form, $y' + P(x)y = Q(x)$, before solving.

$$\begin{aligned} xy' &= y + 3x^2 \sin x \\ \iff y' &= \frac{y}{x} + 3x \sin x \\ \iff y' - \frac{1}{x}y &= 3x \sin x \end{aligned}$$

So we define our integrating factor as

$$I(x) = e^{\int P(x)\,\mathrm{d}x} = e^{-\int \frac{1}{x}\,\mathrm{d}x} = e^{-\ln|x|} = x^{-1} = \frac{1}{x}$$

Then we multiply both sides of the D.E. by the integrating factor, and by the chain rule,

$$\begin{aligned} \frac{1}{x} \cdot \left( y' - \frac{1}{x}y \right) &= (3x \sin x) \cdot \frac{1}{x} \\ \iff \frac{\mathrm{d}}{\mathrm{d}x} \cdot \left( \frac{1}{x}y \right) &= 3 \sin x \\ \iff \frac{1}{x}y &= 3 \int \sin x \,\mathrm{d}x \\ \iff \frac{1}{x}y &= -3\cos x + C \\ \iff y &= -3x \cos x + Cx \end{aligned}$$

Now utilizing the initial condition $y(\pi) = 0$, we have that

$$\begin{aligned} y(\pi) &= -3\pi\cos\pi + C\pi = 0 \\ \iff 3\pi + C\pi &= 0 \\ \iff C &= -3 \end{aligned}$$

So the particular solution is $\boxed{y = -3x\cos x - 3x}$.

11. Solve the initial value problem:

$$xy' = y^2,\ y(1) = 1$$

**Solution:** This is definitely separable; see,

$$\begin{aligned} xy' &= y^2 \\ \iff x\frac{\mathrm{d}y}{\mathrm{d}x} &= y^2 \\ \iff y^{-2}\,\mathrm{d}y &= x^{-1}\,\mathrm{d}x \\ \iff \int y^{-2}\,\mathrm{d}y &= \int x^{-1}\,\mathrm{d}x \\ \iff -\frac{1}{y} &= \ln|x| + C_0 \\ \iff \frac{1}{y} &= C - \ln|x| \qquad C = -C_0 \\ \iff y &= \frac{1}{C - \ln|x|} \end{aligned}$$

Now we use the initial condition $y(1) = 1$ to get the particular solution,

$$\boxed{y = \frac{1}{1 - \ln|x|}}$$

12. Find an equation ($y = ...$) of the tangent to the curve at the given point.

$$x = \cos t + \cos 2t,\ y = \sin t + \sin 2t,\ (x, y) = (-1, 1)$$

**Solution:** Notice that we were given the Cartesian coordinate $(-1, 1)$. Since our curve is defined parametrically, we will need to see at what time $t$ our curve passes through this point. So, we set $x = -1$ and $y = 1$

$$\begin{aligned} &\cos t + \cos 2t = -1 \\ \iff &\cos t + 2\cos^2 t - 1 = -1 \\ \iff &2\cos^2 t + \cos t = 0 \\ \iff &\cos t(2\cos t + 1) = 0 \\ \iff &\cos t = 0 \text{ or } \cos t = -\frac{1}{2} \\ \iff &t = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2} \end{aligned}$$

For the $y$-coordinate, it is perhaps fastest to check if $y = 1$ at any of the previous values of $t$. After checking the four possible values of $t$, we get that that $(x, y) = (-1, 1) \iff t = \pi/2$.

So now that we have the time that our curve passes through $(-1, 1)$, we can find the equation of the tangent line. First, we must find the slope, $\frac{dy}{dx}$ :

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{\cos t + 2\cos 2t}{-\sin t - 2\sin 2t} \end{aligned}$$

At the time $t = \pi/2$, we have

$$\left.\frac{dy}{dx}\right|_{t=\pi/2} = \left.\frac{\cos t + 2\cos 2t}{-\sin t - 2\sin 2t}\right|_{\pi/2} = \frac{-2}{-1} = 2$$

So now, we can use the point-slope form of the line:

$$\begin{aligned} &y - y_1 = m(x - x_1) \\ \implies &y - (1) = 2(x - (-1)) \\ \implies &y - 1 = 2x + 2 \\ \implies &\boxed{y = 2x + 3} \end{aligned}$$

13. Find the Taylor series polynomial of order 3 for the function $y = \sinh(x)$ anchored at $x_0 = 3$.

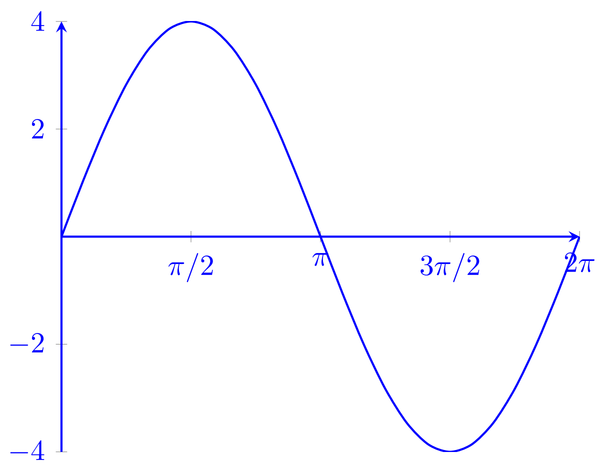
**Solution:**

$$\sinh(x) \approx \sinh(3) + \frac{d}{dx}\sinh(x)\Big|_{x=3}\frac{(x-3)}{2!} + \frac{d^2}{dx^2}\sinh(x)\Big|_{x=3}\frac{(x-3)^2}{2!} + \frac{d^3}{dx^3}\sinh(x)\Big|_{x=3}\frac{(x-3)^3}{3!}$$
$$= \boxed{\sinh(3) + \cosh(3)(x-3) + \sinh(3)\frac{(x-3)^2}{2} + \cosh(3)\frac{(x-3)^3}{6}}$$

14. Let $\quad \boldsymbol{r = f(\theta) = 4\sin(\theta)}$

**(A)** Sketch the graph of $r = f(\theta)$ for $0 \le \theta \le 2\pi$ in CARTESIAN coordinates and identify ALL minima and maxima.

**Solution:**




We can see from this that a maximum occurs at $\theta = \pi/2$ and a minimum occurs at $\theta = 3\pi/2$.

**(B)** Using (A) sketch the graph of $r = f(\theta)$ in POLAR.

**Solution:**

The graph should be a circle of radius 2, repeating every $\pi$ radians, centered at $(x, y) = (0, 2)$. The top, bottom, and sides of the circle should correspond to the horizontal and vertical tangents you find in the following steps.

(C)-(D): Use the fact that the slope in parametric is: $m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$.

**(C)** Find all $(r, \theta)$ points where the curve as a HORIZONTAL tangent.

**Solution:** Since $r = 4\sin\theta$, then $x = 4\sin\theta\cos\theta$ and $y = 4\sin^2\theta$. Then we have that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{8\sin\theta\cos\theta}{4\cos^2\theta - 4\sin^2\theta}$$

A horizontal tangent will occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, so,

$$\begin{aligned} \frac{8\sin\theta\cos\theta}{4\cos^2\theta - 4\sin^2\theta} &= 0 \\ \iff 8\sin\theta\cos\theta &= 0 \\ \iff \theta &= 0, \pi/2, \pi, 3\pi/2, 2\pi \end{aligned}$$

So we have

$$\begin{aligned} r(0) &= 0 \\ r(\pi/2) &= 4 \\ r(\pi) &= 0 \\ r(3\pi/2) &= -4 \\ r(2\pi) &= 0 \end{aligned}$$

$$\boxed{(0,0), (4, \pi/2), (0, \pi), (-4, 3\pi/2), (0, 2\pi)}$$

**(D)** Find all $(r, \theta)$ points where the curve as a VERTICAL tangent.

**Solution:** A vertical tangent will occur when our derivative is undefined, i.e., the denominator of our slope is 0. So,

$$\begin{aligned} 4\cos^2\theta - 4\sin^2\theta &= 0 \\ \iff 4\cos^2\theta &= 4\sin^2\theta \\ \iff 1 &= \tan^2\theta \\ \iff \theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

So we have

$$r\left(\frac{\pi}{4}\right) = \frac{4}{\sqrt{2}}$$
$$r\left(\frac{3\pi}{4}\right) = \frac{4}{\sqrt{2}}$$
$$r\left(\frac{5\pi}{4}\right) = -\frac{4}{\sqrt{2}}$$
$$r\left(\frac{7\pi}{4}\right) = -\frac{4}{\sqrt{2}}$$

$$\boxed{\left(\frac{\pi}{4}, \frac{4}{\sqrt{2}}\right), \left(\frac{3\pi}{4}, \frac{4}{\sqrt{2}}\right), \left(\frac{5\pi}{4}, -\frac{4}{\sqrt{2}}\right), \left(\frac{7\pi}{4}, -\frac{4}{\sqrt{2}}\right)}$$

Note: You may also write $\frac{4}{\sqrt{2}} = 2\sqrt{2}$, which is certainly cleaner.

**(E)** By transforming coordinates from polar → Cartesian, find a CARTESIAN equation for this curve.

**Solution:**

Since $x = r\cos\theta$ and $y = r\sin\theta$, then $x^2 + y^2 = r^2$. We will use this in the following steps:

$$r = 4\sin\theta$$
$$r^2 = 4r\sin\theta$$
$$x^2 + y^2 = 4y$$
$$x^2 + y^2 - 4y = 0$$
$$x^2 + (y-2)^2 - 4 = 0$$
$$x^2 + (y-2)^2 = 4$$

This is a circle centered at $(0, 2)$ with radius 2, as we saw from the plot.

$$\boxed{x^2 + (y-2)^2 = 4}$$

15. Determine the convergence of the following integral. If convergent, compute its value.
$$\int_0^3 \frac{2}{3-x}\,dx$$
**Solution:**

$$\boxed{\text{Diverges}}$$

16. Determine the convergence of the following integral. If convergent, compute its value.
$$\int_0^3 \frac{2}{(3-x)^2}\,dx$$
**Solution:**

$$\boxed{\text{Diverges}}$$

17. Determine the convergence of the following integral. If convergent, compute its value.
$$\int_4^\infty \frac{2}{(3-x)}\,dx$$
**Solution:**

$$\boxed{\text{Diverges}}$$

18. Find the radius of convergence and interval of convergence of the series.

(a) $\displaystyle\sum_{n=0}^{\infty} 4^n \frac{(x-2)^n}{n!}$

**Solution:**

$$\boxed{\text{Radius of Convergence} = \infty}$$

$$\boxed{\text{Interval of Convergence} = (-\infty, \infty)}$$

(b) $\displaystyle\sum_{n=0}^{\infty} (-1)^{n+1}\left(n^2+1\right)\frac{(x+5)^n}{3^n}$

**Solution:**

$$\boxed{\text{Radius of Convergence} = 3}$$

$$\boxed{\text{Interval of Convergence} = (-8, -2)}$$