ACTIVITY#6 — Math 151 — Calculus II — Spring 2021

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NAME (printed): ______________________ ______________________ **Partners:** ______________________________

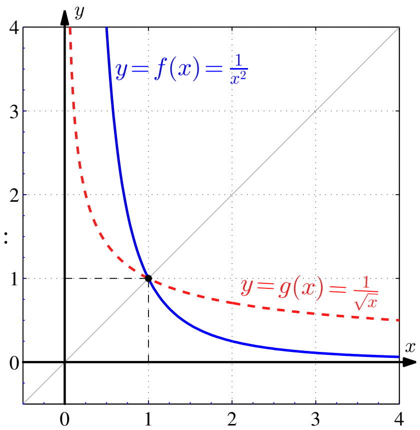
(Family Name) **(First Name)**

Improper Integrals. Let us learn some techniques to check whether some integrals yielding areas with an infinite base (or height), i.e. *improper* integrals, converge (i.e., have finite area) or not.

(1) Consider the following functions: $\boxed{f(x) = \dfrac{1}{x^2}}$ and $\boxed{g(x) = \dfrac{1}{\sqrt{x}}}$. Note that f and g are inverses. $[f^{-1} = g$ and $g^{-1} = f]$

Consider the integrals $I_1 = \displaystyle\int_1^\infty f(x)\,dx$ and $J_1 = \displaystyle\int_0^1 g(x)\,dx$.

(a$_1$) Use the graph to shade these areas and label using the symbols I_1 and J_1.



(b$_1$) From the symmetry of the graph, complete this equation: $J_1 = I_1 + \boxed{}$

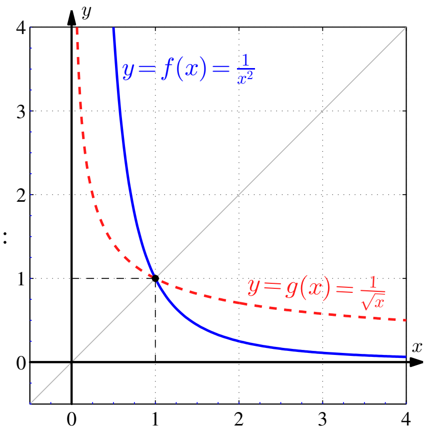
(c$_1$) Compute (using limits):

$I_1 =$

$J_1 =$

Consider now $I_2 = \displaystyle\int_1^\infty g(x)\,dx$ and $J_2 = \displaystyle\int_0^1 f(x)\,dx$.

(a$_2$) Use the graph to shade these areas and label using the symbols I_2 and J_2.



(b$_2$) From the symmetry of the graph, complete this equation: $J_2 = I_2 + \boxed{}$

(c$_2$) Compute (using limits):

$I_2 =$

$J_2 =$

(2) The p-test:

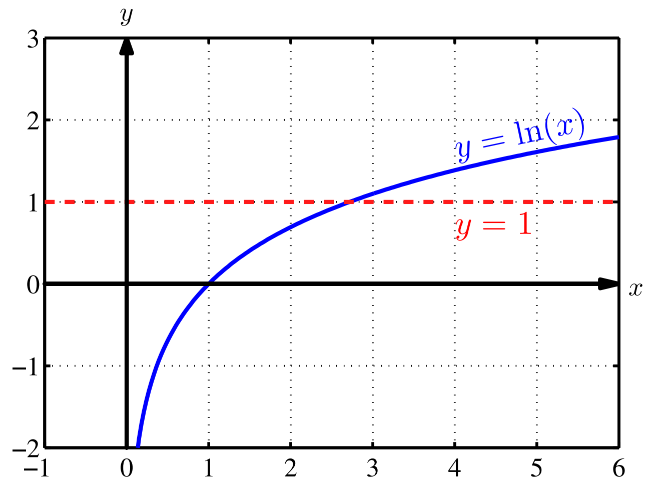
(a) Compute, for both $p \neq 1$ and $p = 1$, the following integrals:

$\underline{p \neq 1}$: $\displaystyle\int \frac{1}{x^p}dx = \int x^{\boxed{}}\,dx = \boxed{} \cdot \frac{1}{x^{\boxed{}}}$ $\underline{p = 1}$: $\displaystyle\int \frac{1}{x}dx = \boxed{}$

(b) Punchline: for which values of p do the improper integrals $\displaystyle\int_a^\infty \frac{1}{x^p}dx$ $(a > 0)$ and $\displaystyle\int_0^b \frac{1}{x^p}dx$ $(0 < b < 1)$ converge? Use DIV for divergent and CONV for convergent:

	Type I	Type II
$p < 1$	$\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$:	$\displaystyle\int_0^b \frac{1}{x^p}\,dx$:
$p = 1$	$\displaystyle\int_a^\infty \frac{1}{x}\,dx$:	$\displaystyle\int_0^b \frac{1}{x}\,dx$:
$p > 1$	$\displaystyle\int_a^\infty \frac{1}{x^p}\,dx$:	$\displaystyle\int_0^b \frac{1}{x^p}\,dx$:

(3) Goal: Is $\int_3^\infty \frac{\ln(x)}{\sqrt{x}} dx$ convergent? [i.e., does it have finite area?]



(a) • Compare $\ln(x)$ with 1. Which one is larger when $x \geq 3$?

• Then what can you say when comparing $\frac{\ln(x)}{\sqrt{x}}$ with $\frac{1}{\sqrt{x}}$? Which one is larger when $x \geq 3$?

• Thus which of the following integrals is larger?

$I_1 = \int_3^\infty \frac{\ln(x)}{\sqrt{x}} dx$ and $I_2 = \int_3^\infty \frac{1}{\sqrt{x}} dx$.

(b) Compute $I_2 = \int_3^\infty \frac{1}{\sqrt{x}} dx =$

(c) What can you say then about $I_1 = \int_3^\infty \frac{\ln(x)}{\sqrt{x}} dx$?

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(4) The punchline:

> **Comparison Theorem for Integrals:**
> If f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$, then
> (a) If $\int_a^\infty f(x)$ is convergent then $\int_a^\infty g(x)$ is ________________.
> (b) If $\int_a^\infty g(x)$ is divergent then $\int_a^\infty f(x)$ is ________________.

(a) If we know that $\int_a^\infty f(x)\,dx$ diverges, what can we say about $\int_a^\infty g(x)\,dx$: convergent / divergent / can't say

(b) If we know that $\int_a^\infty g(x)\,dx$ converges, what can we say about $\int_a^\infty f(x)\,dx$: convergent / divergent / can't say

(c) Why do we need both functions to be positive in the above theorem?

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(5) Consider the integral $\int_2^\infty \frac{\cos^2(x)}{x^2} dx$.

(a) Without computing any integral: Do you think this improper integral converges or diverges? Why?

(b) What is a good comparison function? Write down the inequality and justify your answer in (a)!