ACTIVITY#08 — Math 151 — Calculus II — Spring 2021

Professor/TA: ______________________ **Sec/Time:** ______________ **RedID:** _____________

NAME (printed): _______________ _______________ **Partners:** ______________________________

(Family Name) **(First Name)**

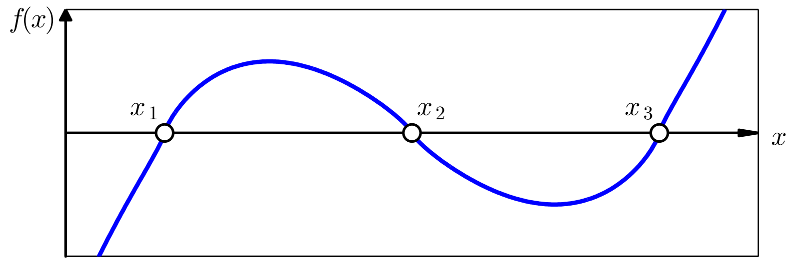
Qualitative analysis of Differential Equations (DEs): The purpose of this activity is to learn how to analyze important features of the dynamics of first order (autonomous) differential equations *without the need to integrate (solve) them!*.
Note: a little birdy told me that this material will be tested in a midterm and/or the final...

---

(1) Consider the following differential equation (DE):

$$\frac{dx}{dt} = f(x), \tag{1}$$

where the sketch of $f(x)$ is given on the figure to the right and where $x(t)$ describes the evolution of x in time (i.e., a trajectory). As the figure shows, $f(x)$ has three zeros (roots): x_1, x_2, and x_3.



(a) Discuss with your partner(s) what is the solution to the DE (1) if the initial condition $x(t\!=\!0)$ is one of the zeros of $f(x)$. These points are called *fixed points*. Why?

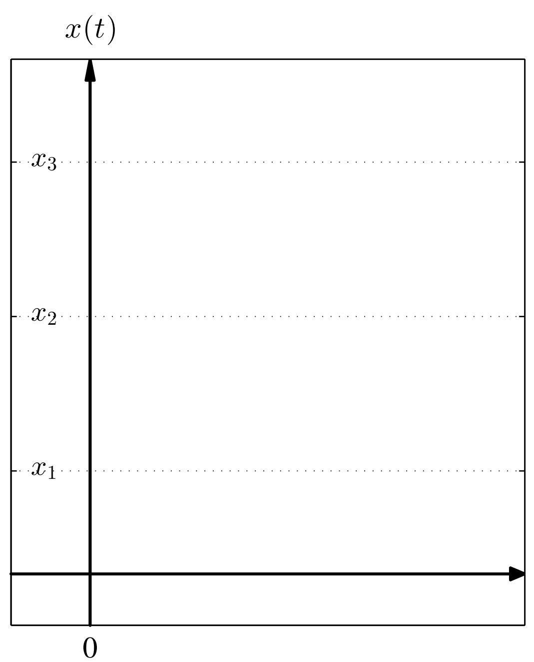
(b) Find the intervals where $f(x) > 0$ and $f(x) < 0$. What happens to $x(t)$ on intervals where $f(x) > 0$? What happens to $x(t)$ on intervals where $f(x) < 0$?

(c) Using the answers on the previous question, draw in the graph above, arrows on the x axis indicating whether $x(t)$ is increasing or decreasing. If increasing: draw a right arrow and if decreasing: draw a left arrow.

(d) What happens to the solution if one starts slightly to the right (or to the left) of x_1? Such a point is called an UNSTABLE fixed point. Why?

What happens to the solution if one starts slightly to the right (or to the left) of x_2? Such a point is called a STABLE fixed point. Why?

(e) In the graph below plot all qualitatively different solutions (trajectories) for initial conditions below/at/above the points x_1, x_2, and x_3.



(f) Can you characterize the stability of the fixed points x_1, x_2, and x_3 by thinking about the slope of f at those points? Be as precise as possible.

The final punchline:

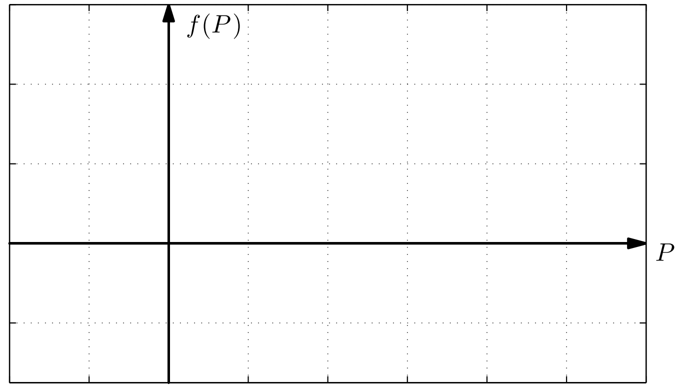
Fixed points and their stability for $dx/dt = f(x)$

- A fixed point x^* is a point such that: $f(x^*) =$ ____.
- A fixed point x^* is **STABLE** if: $f'(x^*)$ ____ 0.
- A fixed point x^* is **UNSTABLE** if: $f'(x^*)$ ____ 0.

(2) Equipped with the experience you gained with the previous problem, let us analyze the so-called **logistic growth** population model:

$$\frac{dP}{dt} = f(P) = rP\left(1 - \frac{P}{M}\right), \tag{2}$$

where r and M are positive **constants**. Your TA will explain how this model is the simplest one that accounts for finite resources.



(a) Draw the polynomial $f(P)$. Be sure to find and label its roots.
Roots:

(b) Draw arrows on indicating where P is increasing/decreasing.

(c) What is the behavior of the logistic model in the long run ($t \to +\infty$) if one starts with an initial $P(t{=}0) = P_0$ such that:

(i) P_0 is precisely at one of the roots:

(ii) P_0 is just to the right of the first root:

(iii) P_0 is to the right of the last root:

(d) (i) What happens to initial conditions very close to each one the fixed points? Do they get "attracted" or "repelled" away from them?

(ii) Why is M called the carrying capacity?

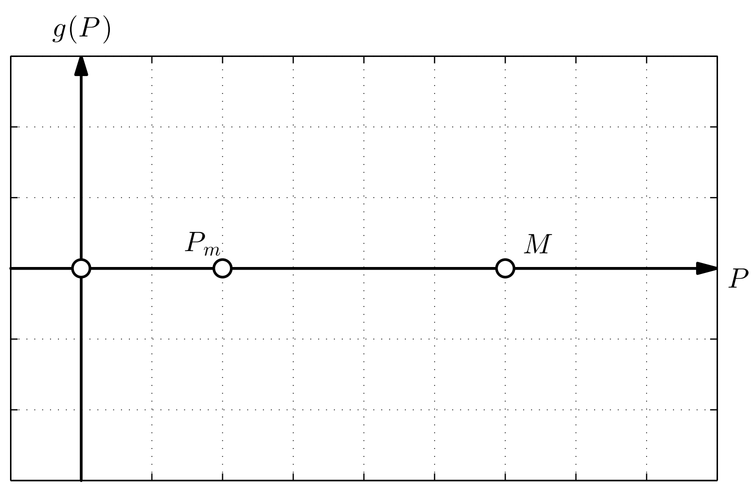
(iii) Why is logistic growth equivalent to exponential (a.k.a. Malthusian) growth [$dP(t)/dt = k\,P(t)$] for *small* populations? Small with respect to what?

---

(3) Application: In many ecological systems, when the population drops below a certain level, it cannot support itself anymore. For example this happens when mates cannot be found, lack of cooperation, being too exposed to predators, etc. This effect is commonly referred to as the **Allee Effect**. A simple model for the Allee effect can be cast in the following form:

$$\frac{dP}{dt} = g(P) \qquad \text{where} \qquad g(P) = -a\,P\,(P - P_m)(P - M), \tag{3}$$

where $a > 0$ is a constant [related to r in Eq. (2)], M is the carrying capacity as before, and P_m is another special population size such that $P_m < M$ (as depicted in the plot).



(a) (i) What are the roots of $g(P)$?

______ , ______ , ______

(ii) $\lim_{P\to\infty} g(P) =$, $\lim_{P\to-\infty} g(P) =$

(iii) Sketch $g(P)$. (Use the limits!)

(iv) Draw, as before, the direction in which the trajectories move on the P-axis.

(v) What can you say, from the sketch, about the stability of the fixed points:

$P_1^* = 0$ is ____________________.

$P_2^* = P_m$ is ____________________.

$P_3^* = M$ is ____________________.

(b) Now, use the punchline in (1)(f), together with the graph of $g(P)$, to corroborate the stability of the fixed points:

$g'(P_1^*)$ __ $0 \Rightarrow P_1^*$ is _______________. $g'(P_2^*)$ __ $0 \Rightarrow P_2^*$ is _______________. $g'(P_3^*)$ __ $0 \Rightarrow P_3^*$ is _______________.

(c) Write an interpretation of the model for this Allee effect in terms of P_m and M. What is so special about P_m?