Cheat Sheet for FINAL - M252 - Calculus III - Fall 2022

This cheat sheet will be included in the midterm. You do NOT have to memorize these formulas. However, make sure that you understand and know how to apply ALL of them!!!

- If a constant is not defined (like an α or any other symbol), do NOT assign a value for it. It is a fixed scalar (or vector) and you should just leave it as it is.
- When dealing with functions of several variables, be careful about which variable you are doing the derivative (or integral) with respect to! [Remember the: "Integral (or derivative) with respect to what?"]
- If $\vec{r} = \langle x, y, z \rangle$ then its norm (magnitude) is $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$
- Making a vector unitary: $\widehat{a} = \dfrac{\vec{a}}{|\vec{a}|}$ ‖ $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos(\theta)$ ‖ $\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ ‖ $\text{proj}_{\vec{a}}(\vec{b}) = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\,\widehat{a}$.
- $\vec{a} \times \vec{b} = |3{\times}3 \text{ determinant}|$ (sign $-$ for middle entry!) ‖ $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\sin(\theta)$
- $|\vec{a} \times \vec{b}|$ = area of parallelogram (= 2×area of triangle) ‖ $|\vec{a} \cdot (\vec{b} \times \vec{c})|$ = volume of parallelepiped.
- Tangent vector: $\vec{T} = \vec{r}\,'(t)$, unitary: $\widehat{T} = \vec{r}\,'(t)/|\vec{r}\,'(t)|$.
- If $|\vec{r}(t)|$ =const then $\vec{r}\,' \perp \vec{r}$.
- Arclength: $L = \int_{t_1}^{t_2} |\vec{r}\,'(t)|\,dt = \int_{t_1}^{t_2} \sqrt{f'^2 + g'^2 + h'^2}\,dt$ ‖ $s(t) = \int_{t_1}^{\tau} |\vec{r}\,'(\tau)|\,d\tau$.
- Curvature in 3D: $\kappa = 1/\text{radius} = \left|\dfrac{d\widehat{T}}{ds}\right| = \left|\dfrac{\widehat{T}\,'}{\vec{r}\,'}\right| = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$.
- Curvature for planar curve $y = F(x)$ in 2D: $\kappa = |F''|/\left(1 + F'^{\,2}\right)^{3/2}$.
- Normal & binormal: $\widehat{N} = \vec{T}\,'/|\vec{T}\,'|$ ‖ $\widehat{B} = \widehat{T} \times \widehat{N}$.
- $\vec{v}(t) = \int \vec{a}(t) + \vec{v}(0)$ ‖ $\vec{r}(t) = \int \vec{v}(t) + \vec{r}(0)$ ‖ $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.
- Tangential and normal components of the acceleration:
  $\vec{a} = \upsilon'\,\widehat{T} + \kappa\upsilon^2\,\widehat{N} = a_T\,\widehat{T} + a_N\,\widehat{N}$ ‖ $\upsilon = |\vec{r}\,'|$ ‖ $a_T = |\vec{r}\,' \cdot \vec{r}\,''|/|\vec{r}\,'|$ ‖ $a_N = |\vec{r}\,' \times \vec{r}\,''|/|\vec{r}\,'|$

- Clairant's theorem: If f, f_{xy}, and f_{yx} are continuous $\Rightarrow$ $f_{xy} = f_{yx}$.
- Tangent plane to $f(x, y)$ at $z_0 = f(x_0, y_0)$: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
  $\Rightarrow$ Linear approximation: $z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
  $\Rightarrow$ Differentials: $dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$
- Chain rule (case 1): $f = f(x, y)$ and $x = x(t)$ and $y = y(t)$: $\dfrac{df}{dt} = \dfrac{dx}{dt}\dfrac{\partial f}{\partial x} + \dfrac{dy}{dt}\dfrac{\partial f}{\partial y}$.
- Chain rule (case 2): $f = f(x, y)$ and $x = x(s, t)$ and $y = y(s, t)$:
  $\dfrac{df}{ds} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$ and $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$
- Implicit differentiation: $F(x, y) = 0$: $\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}$ ‖ $F(x, y, z) = 0$: $\dfrac{dz}{dx} = -\dfrac{F_x}{F_z}$ & $\dfrac{dz}{dy} = -\dfrac{F_y}{F_z}$.
- Directional derivative along $\widehat{u} = \langle a, b \rangle$: $D_{\widehat{u}} f(x, y) = f_x(x, y)\,a + f_y(x, y)\,b = \nabla f \cdot \widehat{u}$.
- Gradient: Nabla operator $= \nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ ‖ gradient$(f) = \nabla f(x, y, z) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle f = \langle f_x, f_y, f_z \rangle$.
- Direction of steepest ASCENT is given by gradient $\nabla f(x, y, z)$ ‖ Gradient vector $\perp$ level curves.
- Tangent plane to $F(x, y, z) = k$ at (x_0, y_0, z_0): $(x - x_0)\,F_x + (y - y_0)\,F_y + (z - z_0)\,F_z = 0$ [with partials evaluated at (x_0, y_0, z_0)].
- Normal line to $F(x, y, z) = k$ at (x_0, y_0, z_0): $\dfrac{x - x_0}{F_x} = \dfrac{y - y_0}{F_y} = \dfrac{z - z_0}{F_z}$ [with partials evaluated at (x_0, y_0, z_0)].
- Remember difference between local and global max/min ‖ For max/min also check along boundaries!
- Max/min will be at critical points: $f_x = 0 = f_y$ (also check boundaries!)
- Second derivative test: $f_x = 0 = f_y$ and Hessian: $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$:
  (a) $D > 0$ & $f_{xx} > 0 \Rightarrow$ local min ‖ (b) $D > 0$ & $f_{xx} < 0 \Rightarrow$ local max ‖ (c) $D < 0 \Rightarrow$ saddle.

- Lagrange multiplier: min/max of f with constraint $g = k$: $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ & $g(x,y,z) = k$.
- Fubini's theorem: if domain is rectangular $\Rightarrow \int_{x=a}^{b}\int_{y=c}^{d} f\,dy\,dx = \int_{y=c}^{d}\int_{x=a}^{b} f\,dx\,dy$.
- Average: $\bar{f} = \frac{1}{A(\mathcal{D})}\iint_{\mathcal{D}} f(x,y)\,dA$.
- Area using double integral: Area($\mathcal{D}$)$=\iint_{\mathcal{D}} 1\,dA$.
- Type I: $\iint_{\mathcal{D}} f(x,y)\,dA = \int_{x=a}^{b}\int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$.
- Type II: $\iint_{\mathcal{D}} f(x,y)\,dA = \int_{y=c}^{d}\int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$.
- Polar: $\iint_{\mathcal{D}} f(x,y)\,dA = \iint_{\mathcal{D}} f(r\cos(\theta), r\sin(\theta))\,\boxed{r}\,dr\,d\theta$. ‖ $r^2 = x^2+y^2$, $x = r\cos(\theta)$, $y = r\sin(\theta)$.
- Cylindrical: $\iiint_{\mathcal{B}} f(x,y,z)\,dV = \iiint_{\mathcal{B}} f(r\cos(\theta), r\sin(\theta), z)\,\boxed{r}\,dr\,d\theta\,dz$.
  with $r^2 = x^2+y^2+z^2$, $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$.
- Spherical: $\iiint_{\mathcal{B}} f(x,y,z)\,dV = \iiint_{\mathcal{B}} f\left(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi)\right)\,\boxed{\rho^2\sin(\phi)}\,d\rho\,d\theta\,d\phi$.
  with $\rho^2 = x^2+y^2+z^2$, $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.

- Surface area: $S = \iint_{\mathcal{D}} \sqrt{f_x^2 + f_y^2 + 1}\,dA$.
- Volume: $V(\mathcal{B}) = \iiint_{\mathcal{B}} dV$
- Mass: $M(\mathcal{B}) = \iiint_{\mathcal{B}} \rho(x,y,z)\,dV$ where $\rho =$ density (similar in 2D).
- Change of variables $(x,y,z) \to (u,v,w)$: $\iiint f(x,y,z)\,dV = \iiint f(u,v,w)\,|J|\,du\,dv\,dw$
  where $J = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$ is the Jacobian of the transformation. [Similar for double integrals]
- Line integrals for *scalar* fields:
  ○ 2D: $\int_{\mathcal{C}} f(x,y)\,dS = \int_{t=a}^{b} f\left(x(t), y(t)\right)\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$ ‖ $x = x(t)$ & $y = y(t)$ for $a \le t \le b$
  ○ 3D (and 2D): $\int_{\mathcal{C}} f(x,y,z)\,dS = \int_{t=a}^{b} f\left(\vec{\boldsymbol{r}}(t)\right)\,|\vec{\boldsymbol{r}}\,'(t)|\,dt$ ‖ $\vec{\boldsymbol{r}}(t) = \langle x(t), y(t), z(t)\rangle$ for $a \le t \le b$
- Line integrals for *vector* fields: $\int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{r}} = \int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{T}}\,dS = \int_{t=a}^{b} \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{r}}\,'\;dt$
- Work: $\vec{\boldsymbol{F}}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$: $W = \int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{r}} = \int_{t=a}^{b} \vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{r}}\,'\;dt = \int P\,dx + Q\,dy + R\,dz$
- Fundamental theorem for line integrals: $\int_{\mathcal{C}} \overrightarrow{\nabla f}\cdot d\vec{\boldsymbol{r}} = f\left(\vec{\boldsymbol{r}}(b)\right) - f\left(\vec{\boldsymbol{r}}(a)\right)$
- A vector field is conservative if it comes from the gradient of a scalar field.
- $\vec{\boldsymbol{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle$ conservative $\Leftrightarrow P_y = Q_x$ [if on a simply-connected domain].
- Green's theorem (GT): $\oint_{\mathcal{C}} P\,dx + Q\,dy = \iint_{\mathcal{D}} (Q_x - P_y)\,dA$ [path must be oriented *anti-clockwise*].
- Punching holes: *inner* paths must be oriented *clockwise* [i.e., *opposite*]. Remember: $\oint_{-\mathcal{C}} = -\oint_{\mathcal{C}}$
- Area using GT: Area enclosed by $\mathcal{C}$: $A = \oint_{\mathcal{C}} x\,dy = \oint_{\mathcal{C}} -y\,dx = \frac{1}{2}\oint_{\mathcal{C}} x\,dy - y\,dx$

- $\text{curl}(\vec{\boldsymbol{F}}) \equiv \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}}$ ‖ $\text{div}(\vec{\boldsymbol{F}}) \equiv \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{F}}$ ‖ $\vec{\boldsymbol{\nabla}}\times(\overrightarrow{\nabla f}) = \vec{\boldsymbol{0}}$ ‖ $\vec{\boldsymbol{\nabla}}\cdot(\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}}) = 0$
- $\vec{\boldsymbol{F}}$ conservative $\Leftrightarrow \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$.
- Surface integral, parametrization $\vec{\boldsymbol{r}} = \langle x(u,v), y(u,v), z(u,v)\rangle$: $\iint_S f(x,y,z)\,dS = \iint_{\mathcal{D}} f(\vec{\boldsymbol{r}}(u,v))\,|\vec{\boldsymbol{r}}_u\times\vec{\boldsymbol{r}}_v|\,du\,dv$
- If S: $z = g(x,y) \Rightarrow \iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + \mathbf{1}}\;dx\,dy$
- Surface area using parametrization ($f = 1$): Area$(S) = \iint_S dS = \iint_{\mathcal{D}} |\vec{\boldsymbol{r}}_u\times\vec{\boldsymbol{r}}_v|\,du\,dv$.
- Surface integral of vector field using parametrization: $\iint_S \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{S}} = \iint_{\mathcal{D}} \vec{\boldsymbol{F}}(u,v)\cdot(\vec{\boldsymbol{r}}_u\times\vec{\boldsymbol{r}}_v)\,du\,dv$ ‖ $d\vec{\boldsymbol{S}} = \widehat{\boldsymbol{n}}\,dS$.
- Green's theorem (vector form, 2D): $\iint_{\mathcal{D}} \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{F}}\,dA = \oint_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot\widehat{\boldsymbol{n}}\,ds$
- Stokes' theorem: $\iint_S \vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{S}} = \oint_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{r}}$ (applies on a "butterfly net" [$\mathcal{C}$=wire and S=net])
- Divergence theorem: $\iiint_E \vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{F}}\,dV = \iint_{\mathcal{S}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{S}}$ (applies on a solid "potato")