M531 - Partial Diff. Eqs (PDEs) — Sp'22 ①

19/Jan/22

State variable : $u(x,t)$ ← space, time $(1+1)D$ ← space, time.

$\underline{2D}|$ $u(x,y,t)$ ⟵ $(2+1)D$

$\underline{3D}|$ $u(x,y,z,t)$ ⟷ $(3+1)D$.

Spatio-temporal systems.

	space	time	variable
Cellular Automata CA	D	D	D
Coupled Map lattices CMLs	D	D	C
ODE	D	C	C
LDE Lattice Diff. Eqs	$D \gg 1$	C	C
PDE	C	C	C

<u>PDE</u> : <u>Def</u> : Eq involving a function on 2 (or more) variables that involves derivatives wrt the variables

$$F\left(u(x,t), \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial t^2}, \ldots\right)$$

$$\boxed{F = m\frac{d^2x}{dt^2}} \to v = \frac{d}{dt}x \to \begin{cases} \frac{dx}{dt} = v(t) \\ \frac{dv}{dt} = F/m \end{cases}$$

$$\frac{\partial u}{\partial t} = G\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right)$$

Notation: $u_x \equiv \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$

$u_t \equiv \frac{\partial u}{\partial t}$, $u_{xxx} = \frac{\partial^3 u}{\partial x^3}$

Nabla operator: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)^T = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$

nabla

"operates" on $u(x,t)$

$$\nabla = (\partial_x, \partial_y, \partial_z)^T$$

* $\nabla f(x,y,z) = (\partial_x, \partial_y, \partial_z) \cdot f(x,y,z)$

$= \partial_x f, \partial_y f, \partial_z f$

$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$ Gradient

* $\nabla \cdot \vec{\phi}(x,y,z) = (\partial_x, \partial_y, \partial_z) \cdot (\underbrace{u, v, w}_{\phi})$

$= \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$ Divergence.

* $\nabla^2 = \nabla \cdot \nabla = (\partial_x, \partial_y, \partial_z) \cdot (\partial_x, \partial_y, \partial_z)$

$= \partial_x^2 + \partial_y^2 + \partial_z^2$

Notation: $\partial_x^2 = \partial_x \partial_x = \partial_{xx}$ || Laplacian

$\nabla^2 f(x,y,z) = (\partial_x^2 + \partial_y^2 + \partial_z^2) \cdot f$

$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

PDE examples

1D: $\nabla = \partial_x$
2D: $\nabla = (\partial_x, \partial_y)$
3D: $\nabla = (\partial_x, \partial_y, \partial_z)$

1| Reaction diffusion, (Heat eq.)

diffusivity

Heat Eq $$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$

x,y,z

More general: $$\frac{\partial u}{\partial t} = Q(x,t) + \nabla \cdot (D(x,y,z) \nabla u)$$

source term

Pop. models $$\frac{d}{dt} n(t) = f(n(t))$$

population

space $\rightsquigarrow$ $n(t) \longrightarrow n(x,t)$

$$\frac{d}{dt} n(x,t) = \underbrace{f(n)}_{} + D \nabla^2 n$$

local term
reaction term

Multiple species
$$\begin{cases} \dfrac{\partial P_1}{\partial t} = f(P_1, P_2) + D_1 \nabla^2 P_1 \\[2ex] \dfrac{\partial P_2}{\partial t} = g(P_1, P_2) + D_2 \nabla^2 P_2 \end{cases}$$

2| Higher order.

$$u_t = -\varepsilon u - 2\nabla^2 u - \nabla^4 u - \underbrace{(\nabla u)^2 - n u^3}_{}$$

Kuramoto-Sivashinsky Eq
$\longrightarrow$ flame propagation.

3| Nonlinear schrödinger eq.

$u(x,y,z,t) \in \mathbb{C} \longrightarrow u = (v + iw)$

$\mathbb{R}$

$$i u_t = -\frac{1}{2} \nabla^2 u + \boxed{|u|^2 u}$$

Nonlinear.

u_1 sol
u_2 sol
$\Rightarrow u_1 + u_2$ is NOT a sol.

Linear Eqs !

If u_1 is solution to Eq.
& u_2 is solution to Eq

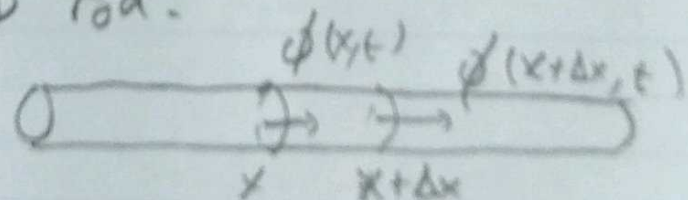
$\Longrightarrow$ $u = \alpha u_1 + \beta u_2$ is ALSO a sol to Eq

arbitrary reals

Gral: $u_t = Q + \underbrace{F(u, u_x, u_{xx}, \dots)}_{\text{LINEAR!}}$

CHAPTER #1 : HEAT EQ.

1.2) Derivation of Heat conduction in (a quasi-) 1D rod.



Def: $e(x,t) =$