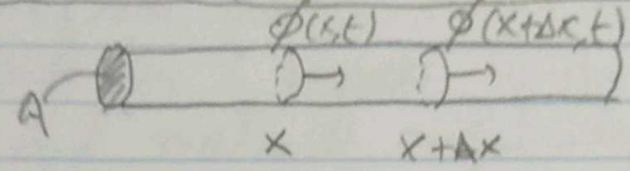
CHAPTER #1: HEAT EQ.

1.2 | Derivation of heat conduction in a (quasi-) 1D rod



$(\Delta x \ll 1)$

Def * $e(x,t) \equiv$ thermal energy density = heat density

$\Rightarrow$ heat energy $= e(x,t) \cdot$ Volume $= e(x,t) \cdot A \cdot \Delta x$

* Energy (heat) conservation:

$\frac{\partial}{\partial t}[eA\Delta x] =$ rate of change of heat energy $=$ $\underbrace{\text{heat lost/gained through boundries}}_{\to \text{flux}}$ $+$ $\underbrace{\text{heat generated or lost inside Volume}}_{\to \text{sinks/sources}}$

* Def: heat flux: $\phi(x)$ = thermal energy per unit time flowing to right per unit surface area

* Def: heat sources: $Q(x,t)$ = thermal energy per unit volume generated per unit time

$\longrightarrow$ inside the Δx-slice the total energy/time is $Q \cdot$ Volume $= Q \cdot A \cdot \Delta x$
[Δx small so $Q(x) \approx$ const]

Conservation: $\frac{\partial}{\partial t}[e\cancel{A}\Delta x] = \phi(x,t)\cdot\cancel{A} - \phi(x+\Delta x,t)\cdot\cancel{A} + Q\cancel{A}\cdot\Delta x$

$\overset{\div\Delta x}{\Longrightarrow}$ $\partial e/\partial t = \dfrac{\phi(x,t) - \phi(x+\Delta x,t)}{\Delta x} + Q$

Now limit $\Delta x \to 0 \Rightarrow$ $\boxed{\partial e/\partial t = -\dfrac{\partial \phi}{\partial x} + Q}$ (1)

We now need to relate $e \longleftrightarrow \phi$

Temp. & specific heat.

Different materials behave differently: the same thermal energy might rise a specific ball of material by 2 degrees while the same ball made of ≠ material might only rise it by 1 degree...

Def: • $u(x,t) \equiv$ temperature

• $c \equiv$ specific heat = heat energy necessary to raise the temp. of a unit mass by one temperature unit

• $\rho(x) \equiv$ material density

∴ In the thin slice we have the energy:

$$\left.\begin{aligned} e\,\cancel{\Delta V} &= c \cdot u \cdot \text{mass} \\ &= c \cdot u \cdot \rho \cdot \Delta V \end{aligned}\right\} \boxed{e(x,t) = c\rho\, u(x,t)}$$

$$\overset{1}{\Longrightarrow} \frac{\partial e}{\partial t} = \frac{\partial}{\partial t}[c \rho u(x,t)] = -\frac{\partial \phi}{\partial x} + Q$$

material properties time INdependent $\left\{\begin{aligned} c &\to c(x) \\ \rho &\to \rho(x) \end{aligned}\right.$

$$\Rightarrow \boxed{c(x)\rho(x)\,\partial u/\partial t = -\partial\phi/\partial x + Q} \quad 2$$

Fourier's law [Equivalent to Hooke's law for springs]

→ relates temp ⟷ flux [$u \leftrightarrow \phi$]

Assume: 1) $u = \text{const} \Longleftrightarrow \phi = 0$

2) ϕ flows from hotter → colder

3) higher temp difference → $\phi \uparrow$

4) ϕ depends of material

The simplest relation that satisfies 1)-4) is a linear relationship called <u>Fourier's law of heat conduction</u>:

$$\phi = -K_0 \, \partial u / \partial x$$

K_0 = material dependent (non-homogeneous material $\longrightarrow$ $K_0 = K_0(x)$)

$$\therefore \quad c(x)\rho(x)\, \partial u/\partial t = -\frac{\partial}{\partial x}\left[-K_0(x)\frac{\partial u}{\partial x}\right] + Q$$

<u>Def</u>: $k(x) \equiv K_0(x)/c(x)\rho(x)$

$$\Longrightarrow \boxed{\frac{\partial u(x,t)}{\partial t} = k(x)\frac{\partial^2}{\partial x^2}u(x,t) + \frac{Q(x,t)}{c(x)\rho(x)}} \quad (3)$$

For homogeneous material and NO sources/sinks

$$\longrightarrow (4) \quad \boxed{\frac{\partial u(x,t)}{\partial t} = k\,\frac{\partial^2 u(x,t)}{\partial x^2}} \quad \text{heat Eq.}$$

↳ This eq. describes the DIFFUSSION of heat. $\longrightarrow$ Diffusion Eq.

↳ It is much more general as long as a similar version to Fourier's law applies for other systems:

- Diffusion of polutants (see text p 9-10)
- Diffusion of population
- etc, etc, ...

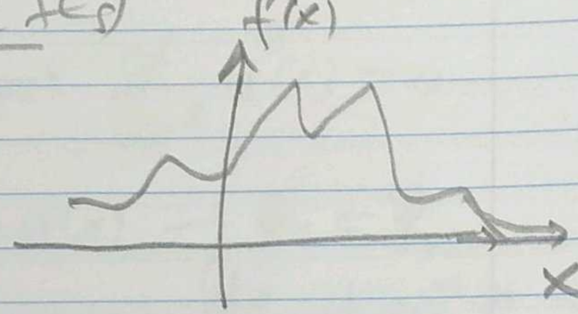
Validity of Diff. eq. relies on fluxes being proportional to gradients of the quantity being modelled.

PDE: $u_t = k\, u_{xx}$, $k > 0$, $u(x,t) \in \mathbb{R}$

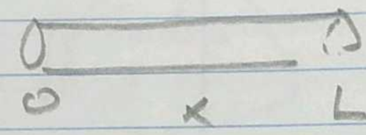
ODE: BVP = ODE + IC
$\uparrow$ Initial conditions.

Initial conditions (ICs)

$\underset{\uparrow t=0}{u(x,0)} = f(x)$



Boundary conditions (BCs)



- Infinite Domain $x \in \mathbb{R}$
- Finite domain $0 \leq x \leq L$

temp. @ edges of rod:

Dirichlet BCs: $\begin{cases} u(x=0,t) = u_0 \\ u(x=L,t) = u_L \end{cases}$ prescribed const. temps.

Neumann BCs: $\begin{cases} \partial u/\partial x \,\big|_{x=0,t} = -\phi_0/K_0 = A_0 \\ \partial u/\partial x \,\big|_{x=L,t} = -\phi_L/K_0 = A_L \end{cases}$

$\rightsquigarrow$ fixing flow rate @ boundaries.

1.4 Equilibrium temp. distribution

1.4.1 prescribed temp @ Boundaries (Dirichlet)

IC: $\boxed{\begin{array}{l} u(x,0) = f(x) \\ u(0,t) = T_1(t) \\ u(L,t) = T_2(t) \end{array} + \quad u_t = k\, u_{xx}}$ (4)

Const. temp. @ Boundaries $\begin{cases} T_1(t) = T_1 \\ T_2(t) = T_2 \end{cases}$

Q: what is the equil. temp. distribution?

A: $\longrightarrow \dfrac{\partial u}{\partial t} = 0 \implies u(x,t) = u^*(x)$

Heat eq. $\implies \cancel{k} u^*_{xx} = 0 \implies \dfrac{\partial^2}{\partial x^2} u^*(x) = 0$

$\cancel{PDE} \to ODE \quad u = u(x) \to \partial_x = \dfrac{d}{dx}$

$\implies \dfrac{d^2}{dx^2} u^*(x) = 0 \xRightarrow{\int dx} \dfrac{d}{dx} u^* = C_1$

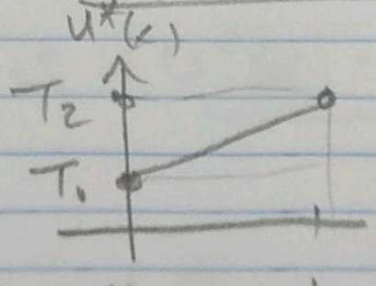
$\xRightarrow{\int dx} \boxed{u^*(x) = C_1 x + C_2}$ || Indep. of BCs

Use BCs $\quad u^*(0) = T_1 \implies C_2 = T_1$

$u^*(L) = T_2 \implies C_1 L + \cancel{C_2} = T_2$

$\implies C_1 = (T_2 - T_1)/L$

So $\boxed{u^*(x) = \dfrac{(T_2 - T_1)}{L} x + T_1}$



$\leftarrow$ Eq. temp for $u_t = k u_{xx}$

$u(0,t) = T_1$

$u(L,t) = T_2$

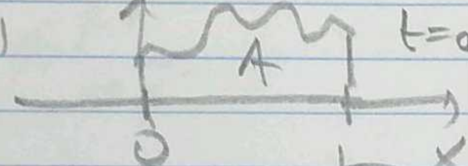
$\cancel{u(x,0) = f(x)}$

1.4.2 Insulated Boundaries.

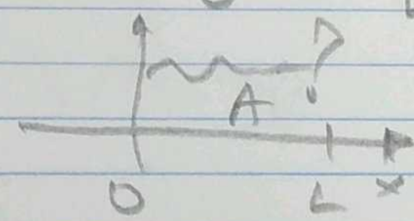
Neumann BCs $\boxed{\dfrac{\partial u(0,t)}{\partial x} = A_0 \qquad \dfrac{\partial u(L,t)}{\partial x} = A_L}$

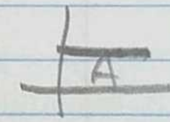
$A_0 = A_L = 0 \Rightarrow$ No flux accross boundries
$\longrightarrow$ INSULATED.

<u>Q:</u> $u(x,0) = f(x)$



$\xrightarrow{t\to\infty}$





<u>Math</u>: From b4 $u^*(x) = C_1 x + C_2$

<u>BCs</u>: $\dfrac{\partial u}{\partial x} = C_1 \xrightarrow{x=0} \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = A_0 = 0 \Rightarrow C_1 = 0$

$\qquad\qquad\xrightarrow{x=L} \left.\dfrac{\partial u}{\partial x}\right|_{x=L} = A_L = 0 \Rightarrow C_1 = 0$

$C_1 = 0 \quad \therefore \quad \boxed{u^*(x) = C_2}$ $\qquad \left[\underbrace{C_2 = \frac{1}{L}\int_0^L f(x)\,dx}_{\text{Physics.}} \right.$

<u>Conservation of Energy</u>

- Physics way $\longrightarrow$ see textbook.
- Math: $u_t = k u_{xx} \overset{\int_0^L dx}{\Longrightarrow} \dfrac{\partial}{\partial t}\int_0^L u\,dx = \cancel{\left[k u_x\right]_0^L}^{\,0}$

but $\left.\begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0 \end{aligned}\right\}$ Insulated.

$$\Rightarrow \frac{\partial}{\partial t} \int_0^L u(x,t)\,dx = 0$$

$$\Rightarrow \int_0^L u(x,t)\,dx = \text{const} = \int_0^L \underbrace{f(x)}_{u(0,t)}\,dx$$

$\therefore$ for equilibrium $u^* = C_2$ -

$$\Rightarrow \int_0^L \underset{\uparrow \text{final}}{u^*}\,dx = \int_0^L \underset{\uparrow \text{initial}}{f(x)}\,dx$$

$$\Rightarrow C_2 \overbrace{\int_0^L dx}^{L} = \int_0^L f(x)\,dx$$

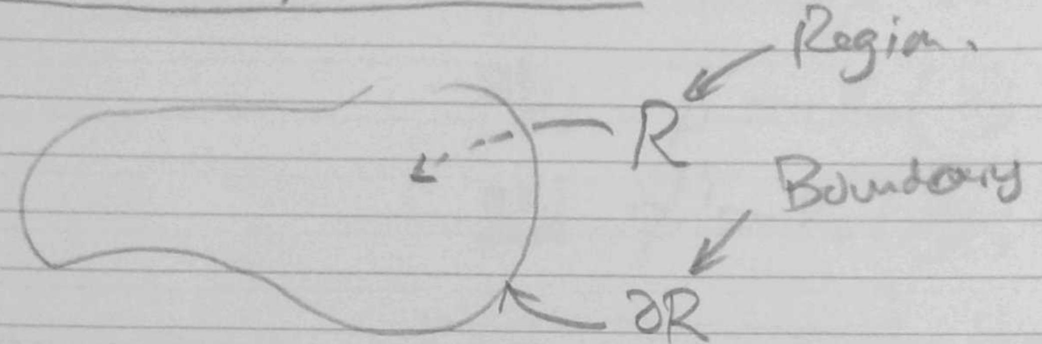
$$\Rightarrow \boxed{C_2 = \frac{1}{L}\int_0^L f(x)\,dx}$$

$\therefore$ If insulated boundaries and ARBITRARY initial temp. distribution $f(x)$

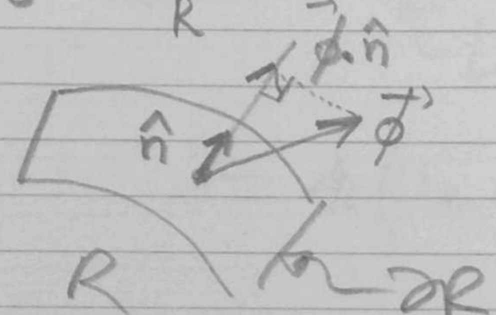
$$u(x,t) \xrightarrow{t\to\infty} u^*(x) = \frac{1}{L}\int_0^L f(x)\,dx$$

M = "mass" = $\int_0^L u(x,t)\,dx$
is a CONSERVED quantity in the heat (diffusion) Eq. when INSULATED boundaries.

1.5 Heat eq. in 2D & 3D



- Total energy $\iiint_R e\,dV = \iiint c(\vec{r})\rho(\vec{r})u(\vec{r},t)\,dV$
- Fluxes: $r=(x,y,z)$



- conservation of E

$$\frac{d}{dt}\iiint_R c\rho u\,dV = -\oint\!\!\oint_{\partial R} \vec{\phi}\cdot\hat{n}\,dS + \iiint_R Q\,dV \quad (5)$$

Divergence theo: NABLA = DEL

Def: $\mathrm{Div}\,\vec{\phi} = \nabla\cdot\vec{\phi} = (\partial_x,\partial_y,\partial_z)\cdot(\phi_x,\phi_y,\phi_z)$
$= \frac{\partial\phi_x}{\partial x} + \frac{\partial\phi_y}{\partial y} + \frac{\partial\phi_z}{\partial z}$

Theo: $\iiint_R \nabla\cdot\vec{\phi}\,dV = \oint\!\!\oint_{\partial R} \vec{\phi}\cdot\hat{n}\,dS$

$$(5) \Rightarrow \iiint_R \left[c\rho\frac{\partial u}{\partial t} + \nabla\cdot\vec{\phi} - Q\right] = 0 \quad (6)$$

Is true $\forall R \rightarrow$ integrand $= 0$

<u>Fourier law</u>

1D: $\phi = -K_0 \, \partial u / \partial x$

2D: $\vec{\phi} = -K_0 \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)$ (7)

3D: $\vec{\phi} = -K_0 \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)$

(7) + (6) $\Rightarrow$ $\boxed{\frac{\partial u}{\partial t} = k \nabla^2 u}$ Heat eq. in ND.

$Q = 0$

no sink no sources

Laplacian. $\nabla \cdot \nabla$.

$= k (u_{xx} + u_{yy} + u_{zz})$