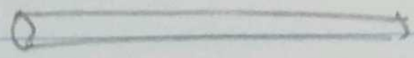
M531 – 26/Jan/22 1/

CHAP 2: Separation of Variables.

Ex:  $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{Q(x,t)}{c\rho}$

& IC: $u(x,0) = f(x)$

BCs: $u(0,t) = T_1(t)$

$u(L,t) = T_2(t)$

Linearity: • In gral an operator L is linear if:

$$L(\alpha u_1 + \beta u_2) = \alpha L(u_1) + \beta L(u_2)$$
$$\alpha, \beta \in \mathbb{R}.$$

• Ex $\partial_x = \frac{\partial}{\partial x} = L$

$$\frac{\partial}{\partial x}[\alpha f(x) + \beta g(x)]$$
$$= \alpha \partial_x f + \beta \partial_x g$$

* $\nabla^2(\alpha u + \beta v) = \alpha \nabla^2 u + \beta \nabla^2 v$

* Heat operator: $\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$

$$\underbrace{\left[\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}\right]}_{\equiv H} u = 0$$

u_1 is sol $\rightarrow H[u_1] = \frac{\partial u_1}{\partial t} - k\frac{\partial^2 u_1}{\partial x^2} = 0$

u_2 is sol $\rightarrow H[u_2] = \cdots = 0$

$\Longrightarrow H[\alpha u_1 + \beta u_2] = 0 \quad \forall \alpha, \beta \in \mathbb{R}$

superposition principle.

- Nonlinear $\nabla^2 u + u^2 = 0$

+cos / tan 1/.
+sin √

$$\left(\frac{\partial u}{\partial x}\right)^2 \neq \frac{\partial^2 u}{\partial x^2}$$

Nonlinear linear..

<u>Homogeneity</u>: zero.

If PDE[u=0] = 0 ⇒ PDE is HOMOGENEOUS

<u>Ex</u>: • $u_t - k u_{xx} = 0$ Heat Eq.
$u = 0 \longrightarrow \; = 0$ ✓ is Homog.

• If sink/sources are present.

$$\underbrace{\overbrace{u_t - k u_{xx}}^{N[u]} - \frac{Q}{c\rho}}_{PDE[u]} = 0 \quad f$$

$u = 0 \Rightarrow PDE[0] = -\frac{Q}{c\rho} \neq 0$ ∴ Non-Homogeneous.

* Usually, split PDE into homogeneous part $N[u]$ & the remainder $f(x,t)$ forcing

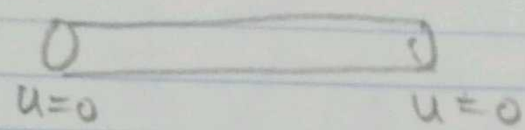
$$PDE[u] = 0 \Rightarrow \underbrace{N[u]}_{\text{homogeneous}} = \underbrace{f}_{\text{Non-homog.}}$$

⚠ the concept of linearity also applies to BCs

<u>BCs</u> : * $u(0,t) = f(t)$ linear & Non-homog.

* $\frac{\partial u}{\partial x}(L,t) = 0$ linear & Homog.

<u>2.3]</u> Heat eq. with temp $= 0$ @ finite ends
[& $Q = 0$]



$$\boxed{\begin{array}{ll} \bullet\ \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} \quad (1) & 0<x<L \\ & t>0 \\ \bullet\ u(x,0) = f(x) & \\ \bullet\ \begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases} & \end{array}}$$

Linear & Homogeneous PDE + BCs

Linear PDEs $\longrightarrow$ SEPARATION of VARIABLES

<u>SEP. VAR.</u> $u(x,t) = \phi(x)\, G(t)$ (2)
assume

$\xrightarrow{PDE}$ $\phi(x) = ?$ & $G(t) = ?$

$$(1) \xrightarrow{(2)} \frac{\partial}{\partial t}[\phi(x) G(t)] = k \frac{\partial^2}{\partial x^2}[\phi(x) G(t)]$$

$$\rightarrow \phi(x) \frac{\partial}{\partial t} G(t) = k\, G(t) \frac{\partial^2 \phi(x)}{\partial x^2}$$

$$\xrightarrow{\div \phi G k} \frac{\cancel{\phi}}{\cancel{\phi} G k} \underbrace{G'(t)}_{\partial t} = \frac{\cancel{\phi} G}{\cancel{\phi} G k} \underbrace{\phi''(x)}_{\partial x}$$

$$\overleftarrow{t\text{ stuff}} \qquad \overrightarrow{x\text{ stuff}}$$

$$\Rightarrow \frac{1}{kG(t)} G'(t) = \frac{1}{\phi(x)} \phi''(x)$$ ← Must be true for all x's & t's

~~ODE~~: $\frac{dx}{dt} = f(x) \cdot g(t)$ (x stuff ← , t-stuff →)

$$\Rightarrow \int \frac{1}{f(x)}\, dx = \int g(t)\, dt$$

$$\text{If} \quad \frac{1}{kG} G' = \frac{1}{\phi} \phi'' = \text{const} = -\lambda$$

↑ separation constant.

$$\Rightarrow \begin{cases} \frac{G'}{kG} = -\lambda \\ \frac{\phi''}{\phi} = -\lambda \end{cases} \quad \Leftrightarrow \text{Heat. eq}$$

$$\Rightarrow \begin{cases} G'(t) = -\lambda k\, G(t) \\ \phi''(x) = -\lambda\, \phi(x) \end{cases} \qquad \lambda, k \in \mathbb{R}, \ k>0$$

→ Solve the G & ϕ eqs and from there force BCs & ICs.

Def: λ = sep. const. = EIGENVALUE of the PDE

Force the BCs: $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$

$u = \phi G$

$$\Rightarrow \begin{cases} \phi(0) \cdot G(t) = 0 \rightarrow \phi(0) = 0 \text{ OR } G(t) = 0 \\ \phi(L) \cdot G(t) = 0 \rightarrow \phi(L) = 0 \text{ OR } G(t) = 0 \end{cases}$$

$G(t) = 0 \Rightarrow u = 0 \rightarrow$ trivial sol.

* $u=0$ does NOT satisfy ICs.

$\therefore G(t)\neq 0 \implies \begin{cases}\phi(0)=0\\ \phi(L)=0\end{cases}$

$\therefore$ SPATIAL $\left|\begin{cases}\phi''(x)=-\lambda\phi(x)\\ \phi(0)=0\\ \phi(L)=0\end{cases}\right| \begin{matrix}\text{ODE}\\ \text{BCs}\end{matrix}$ } BVP boundary value problem.

still to be solved.

TEMPORAL PART:

$G'(t)=-\lambda k\, G(t) \implies \boxed{G(t)=G_0 e^{-\lambda k t}}$

SPATIAL PART

$\phi(x)=A\sin\sqrt{\lambda}x+B\cos\sqrt{\lambda}x$

$\phi''(x)=-\lambda\phi(x)$

GRAL: $\frac{d^n u}{dx^n}+a_{n-1}\frac{d^{n-1}u}{dx^{n-1}}+\dots+a_1\frac{du}{dx}+a_0u=0$

Gral. n-th order, linear ODE.

Characteristic polyn: $z^n+a_{n-1}z^{n-1}+\dots+a_1z+a_0=0$

If all n roots $z_1,\dots,z_n$ are distinct.

$\implies u(x)=\alpha_1 e^{z_1x}+\alpha_2 e^{z_2x}+\dots+\alpha_n e^{z_nx}$

$u(x)=\sum_{i=1}^{n}\alpha_i e^{z_ix}$

Back to $\phi''(x) = -\lambda\phi(x)$

$\hookrightarrow$ Charac. polyn. $z^2 = -\lambda \rightarrow z = \pm i\sqrt{\lambda}$

$$\Rightarrow \phi(x) = \alpha_1 e^{-i\sqrt{\lambda}x} + \alpha_2 e^{+i\sqrt{\lambda}x}$$

$\underline{\lambda > 0}$

$$= \alpha_1 [\cos(-\sqrt{\lambda}x) + i\sin(-\sqrt{\lambda}x)]$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$$+ \alpha_2 [\cos(\sqrt{\lambda}x) + i\sin(\sqrt{\lambda}x)]$$

$$= \underbrace{(\alpha_1 + \alpha_2)}_{A} \cos\sqrt{\lambda}x + i(-\alpha_1 + \alpha_2)\sin\sqrt{\lambda}x$$

+ SPATIAL PART: $\phi'' = -\lambda\phi$

$$\Rightarrow \phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

BCs: $\phi(0) = 0$ & $\phi(L) = 0$

+ TEMPORAL PART $G(t) = G_0 e^{-k\lambda t}$

+ ICs: $u(x,0) = f(x)$

Sep. Var: $u(x,t) = \phi(x)\cdot G(t)$