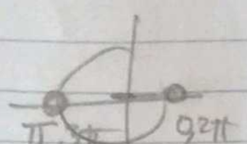
Spatial part: $\phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$+\ \phi(0)=0$ & $\phi(L)=0$

$\phi(0)=0 \Rightarrow A\overset{1}{\cos 0} + B\overset{0}{\sin 0} = 0 \Rightarrow \boxed{A=0}$

$\phi(L)=0 \Rightarrow B\sin\sqrt{\lambda}L = 0 \Rightarrow \boxed{B=0}$

$B\neq 0$ to get any non-trivial sol'



$\sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = n\pi,\ n\in\mathbb{Z}$

$\Rightarrow \boxed{\lambda_n = \frac{n^2\pi^2}{L^2}}\quad n\in\mathbb{Z}$

$\therefore\ \phi(x) = B\sin\underbrace{\frac{n\pi}{L}}_{\sqrt{\lambda_n}}x$

$\phi(x) = \underbrace{B_1\sin\frac{1\pi}{L}x}_{\phi_1} + \underbrace{B_2\sin\frac{2\pi}{L}x}_{\phi_2} + \dots$

Most gral: $\boxed{\phi(x) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{L}x}$

↳ spatial part + BCs

~~$\lambda=0$~~ | $\phi'' = -\lambda\phi \overset{\lambda=0}{=} 0 \overset{\int}{\Rightarrow} \phi' = C$

$\overset{\int}{\Rightarrow} \phi(x) = Cx + D$

BCs

$\phi(0)=\phi(L)=0 \Rightarrow C=D=0$

⚠ $u=0$

$\underline{\lambda<0}$: $\phi'' = -\lambda\phi = s\phi$

$s = -\lambda > 0$ $\qquad \phi = E e^{\sqrt{s}x} + F e^{-\sqrt{s}x}$

$\qquad = G \sinh\sqrt{s}x + H \cosh\sqrt{s}x$

$\underline{BCs}$: • $\phi(0)=0 \Rightarrow G \overset{0}{\sinh 0} + H \overset{1}{\cosh 0} = 0$

$\Rightarrow \boxed{H=0}$

• $\phi(L)=0 \Rightarrow G \underbrace{\sinh\sqrt{s}L}_{>0} = 0 \Rightarrow \boxed{G=0}$

$\underset{H=G=0}{\Longrightarrow} \phi = 0 \Rightarrow u = 0 \Rightarrow ?$

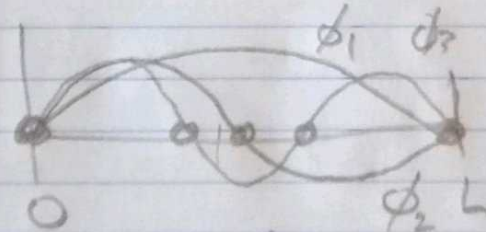
Eigenvalues & Eigenfunctions.

$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad B_n \sin\sqrt{\lambda_n}x = \phi_n$

$\underline{\text{Full (product) solution}}$: $\quad u(x,t) = \phi(x)G(t)$

$\phi(x) = \sum_{n=1}^{\infty} B_n \phi_n = \sum_{n=1}^{\infty} B_n \sin\sqrt{\lambda_n}x, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$

Basis = "eigen basis" = $\left\{\sin\frac{n\pi}{L}x\right\}_{n=1}^{\infty}$



ϕ_n has $(n+1)$ zeros

TEMPLATES

$\Rightarrow u(x,t) = \left(\sum_{n=1}^{\infty} \underline{\underline{B_n}} \sin\frac{n\pi}{L}x\right) \underline{\underline{G_0}} e^{-k\lambda_n t}$

$A_n \equiv B_n C_0$ $\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$

<u>ICs</u>: $u(x,0) = f(x)$

$\Rightarrow \boxed{\underbrace{\sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \, e^{0}}_{\text{FOURIER-SINE SERIES.}} = f(x)}$

$A_n = \{A_1, A_2, A_3, \ldots\}$ need to chosen such that $\sum A_n \phi_n = f(x)$

<u>Orthogonality of eigenfunction</u>

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \quad (1) \longrightarrow A_n = ?$$

<u>orthog.</u>: $$\frac{2}{L}\int_0^L \underbrace{\sin\frac{n\pi}{L}x}_{\phi_n} \cdot \underbrace{\sin\frac{m\pi}{L}x}_{\phi_m} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$$\equiv \langle \phi_n, \phi_m \rangle$$

$$\equiv \phi_n \cdot \phi_m$$

$$\equiv \phi_n \text{ "DOT" } \phi_m$$

$$\phi_n = \left\{\sqrt{\frac{2}{L}} \sin\frac{n\pi}{L}x\right\} \nearrow \langle \phi_n, \phi_m \rangle = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

↑ ORTHONORMAL BASIS

$\int ① \cdot \phi_m$

① $\longrightarrow \int_0^L f(x) \sin\frac{n\pi}{L}x \, dx = \int_0^L \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x \cdot \sin\frac{m\pi}{L}x \, dx$

$= \sum_{n=1}^{\infty} A_n \int_0^L \sin\frac{n\pi}{L}x \, \sin\frac{m\pi}{L}x \, dx \; \begin{cases} L/2 & n=m \\ 0 & m \neq m \end{cases}$

$= A_0 \cdot 0 + A_1 0 + \cdots \overset{n=m}{A_m L/2} + 0 + 0 + 0$

$\Rightarrow \quad A_m \frac{L}{2} = \int_0^L f(x) \sin\frac{m\pi}{L}x \, dx$

$\longrightarrow \boxed{A_m = \frac{2}{L} \underbrace{\int_0^L f(x) \sin\frac{m\pi}{L}x \, dx}_{\langle f, \phi_m \rangle}}$

$\phi_n = \sqrt{\frac{2}{L}} \sin\frac{n\pi}{L}x$

$\boxed{\underbrace{u(x,t) = \sum A_n \sin\frac{n\pi}{L}x \, e^{-k\left(\frac{n\pi}{L}\right)^2 t}}}$

sol to Heat eq. $\frac{\partial u}{\partial t} + k \frac{\partial^2 u}{\partial x^2} = 0$

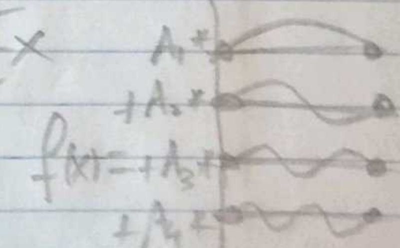
$\propto \; \propto$

BCs $u(0,t) = 0 = u(L,t)$

Ic: $u(x,0) = f(x)$

* spatial templates $\phi_n \propto \sin\frac{n\pi}{L}x$

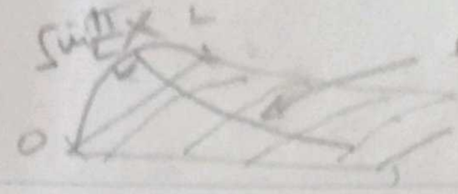
prop to.



* Spatio-temporal template

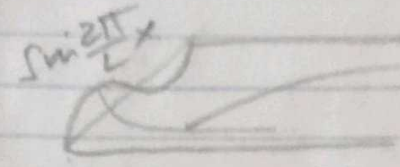
$U_n(x,t) = \sin\frac{n\pi}{L}x \, e^{-k\left(\frac{n\pi}{L}\right)^2 t}$ $+A_5$

$\longrightarrow U(x,t) = \sum A_n U_n(x,t)$

$A_1 u_1 =$ 

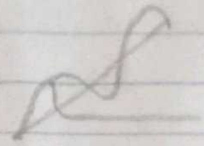
$e^{-k(\frac{\pi}{L})^2 t}$

$+$

$A_2 u_2 =$ 

$e^{-k(\frac{2\pi}{L})^2 t}$

$u =$ $+$

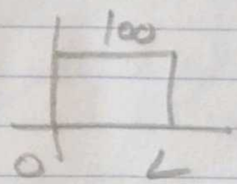
$A_3 u_3 =$ 

$+$

$A_4 u_4 =$

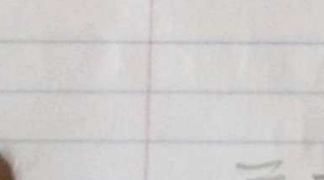
$+$

<u>Ex 2.3.7</u> Heat Eq $+$ $u(0,t) = 0 = u(L,t)$
$+$ $u(x,0) = 100$

<u>t=0</u> 

$\xrightarrow{t}$?

$\rightsquigarrow A_n = ? = \frac{2}{L}\int_0^L \underbrace{100}_{f(x)} \sin\frac{n\pi}{L}x\,dx$



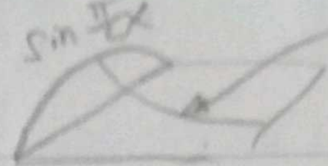
$= \frac{200}{\cancel{L}}\left[-\frac{\cancel{L}}{n\pi}\cos\frac{n\pi}{L}x\right]_0^L = -\frac{200}{n\pi}\left[\quad\right]_0^L$

$= -\frac{200}{n\pi}\left[\underbrace{\cos\frac{n\pi}{\cancel{L}}\cancel{L}}_{(-1)^n} - \overbrace{\cos\frac{n\pi}{L}0}^{1}\right]$

$= -\frac{200}{n\pi}\left[(-1)^n - 1\right]$

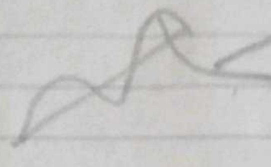
$\Rightarrow \begin{cases} A_{n,even} = 0 \\ A_{n,odd} = 400/n\pi \end{cases}$

$n=1 \Rightarrow A_1 = -\frac{200}{\pi}[-2]$
$A_2 = 0$
$A_3 = 400/3\pi$
$A_4 = 0$
$A_5 = 400/5\pi$
$A_6 = 0$

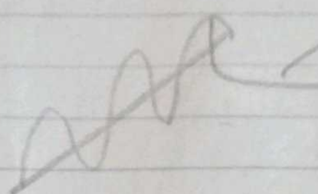
$u = \overbrace{\left(\frac{400}{\pi}\right)}^{A_1} *$ 

$e^{-k\left(\frac{\pi}{L}\right)^2 t}$

$+ \; \overbrace{0}^{A_2}$

$+ \; \overbrace{\frac{400}{3\pi}}^{A_3} *$  $e^{-k\left(\frac{3\pi}{L}\right)^2 t}$

$+ \; \overbrace{0}^{A_4}$

$+ \; \frac{400}{5\pi} *$  $e^{-k\left(\frac{5\pi}{L}\right)^2 t}$

$\vdots$

$=$