Intermission → CHAP 3: FOURIER SERIES

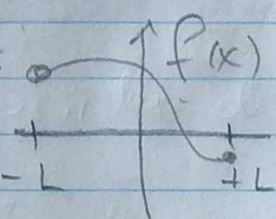
Goal: to express periodic functions in terms of sin | cos | complex exp

Similar to Taylor: * $\vec{V} = \sum_{i=1}^{D} a_i \hat{e}_i$; $a_i = \vec{V} \cdot \hat{e}_i$

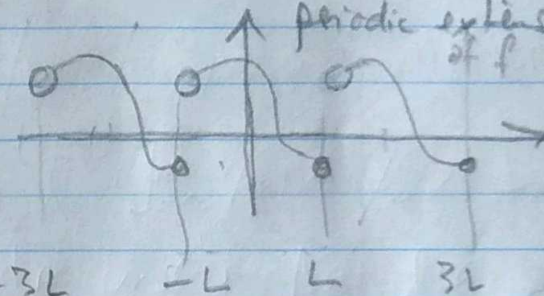
* $f(x) = \sum_{n=0}^{\infty} A_n (x-a)^n$, $A_n = \dfrac{f^{(n)}(a)}{n!}$

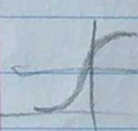
* Here the basis is $\phi_n = \{x^n\}_{n=0}^{\infty} = \{1, x, x^2, x^3, \ldots\}$

Fourier:



periodic $\longrightarrow$ extension on $[-L, L]$





Use projection into sin & cos basis and use the orthogonality of sin's and cos's:

$$\left\langle \cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L} \right\rangle = \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = \begin{cases} 2L & \text{if } n=m=0 \\ L & \text{if } n=m\neq 0 \\ 0 & \text{if } n\neq m \end{cases}$$

$$\left\langle \sin\frac{n\pi x}{L}, \sin\frac{m\pi x}{L} \right\rangle = \int_{-L}^{L} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x\, dx = \begin{cases} L & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$

$$\left\langle \sin\frac{n\pi x}{L}, \cos\frac{m\pi}{L}x \right\rangle = 0, \ \forall n, m$$

[Similar formulas for $\int_0^L$]

Basis: $\{\phi_n\} = \left\{ \sin\frac{n\pi}{L}x, \cos\frac{n\pi}{L}x \right\}_{n=0}^{\infty}$

Expand: $f(x) = \sum c_n \phi_n$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=0}^{\infty} b_n \sin\frac{n\pi}{L}x$$

① $\Rightarrow \boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x}$

Q: $a_0 = ?$, $a_n = ?$, $b_n = ?$

↳ A: use $\perp$ of basis & project!

Project $f(x)$:

(a) $\langle f(x), a_0 \rangle \longrightarrow a_0$

(b) $\langle f(x), a_m \cos\frac{m\pi}{L}x \rangle \longrightarrow a_m$

(c) $\langle f(x), b_m \sin\frac{m\pi}{L}x \rangle \longrightarrow b_m$

(a) $\overset{①}{\Rightarrow}$ $\int_{-L}^{L} f(x)\cdot \cancel{a_0}\, dx = \int (a_0^{\cancel{2}} + \sum \cancel{a_0} a_n C + \sum \cancel{a_0} b_n S)$

$\Rightarrow \int_{-L}^{L} f(x)\, dx = a_0 \int_{-L}^{L} dx + \sum a_n \underbrace{\cancel{\int_{-L}^{L} \cos\frac{n\pi}{L} dx}}_{\text{period}\ \to 0} + \sum b_n \cancel{\int_{-L}^{L} \sin\frac{n\pi}{L}x}^{\,0}$

$= a_0 [x]_{-L}^{L} = a_0(L-(-L)) = 2L a_0$

$\Rightarrow \boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx}$ ← a_0 is average of $f(x)$

(b) $\overset{①}{\Rightarrow} \langle f, a_m C \rangle = \int_{-L}^{L} f(x) a_m C = \int_{-L}^{L} a_0 a_n \cancel{\cos\frac{n\pi}{L}x}^{\,0\ \text{period}}\, dx$

$+ \underbrace{\int_{-L}^{L} \sum_n a_n C_n a_m C_m}_{(n=m)\ a_m^2 \cdot L} + \int_{-L}^{L} \sum_n \cancel{b_n S_n a_m C_m}$ → 0 Sin ⊥ Cos

$\Rightarrow \cancel{a_m} \int_{-L}^{L} f(x) \cos\frac{m\pi}{L}x\, dx = a_m^{\cancel{2}} \cdot L$

$\Rightarrow \boxed{a_m = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{m\pi}{L}x\, dx = \frac{1}{L}\langle f, \cos\frac{m\pi}{L}x \rangle}$

(c) Same as (b) $\longrightarrow$ $\boxed{b_m = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{m\pi}{L}x\, dx}$

$= \frac{1}{L}\langle f(x), \sin\frac{m\pi}{L}x \rangle$

$\therefore$ If $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} dx$

$$\Rightarrow \begin{cases} a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx \\ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx \\ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \end{cases}$$

Sin-cos Fourie series.

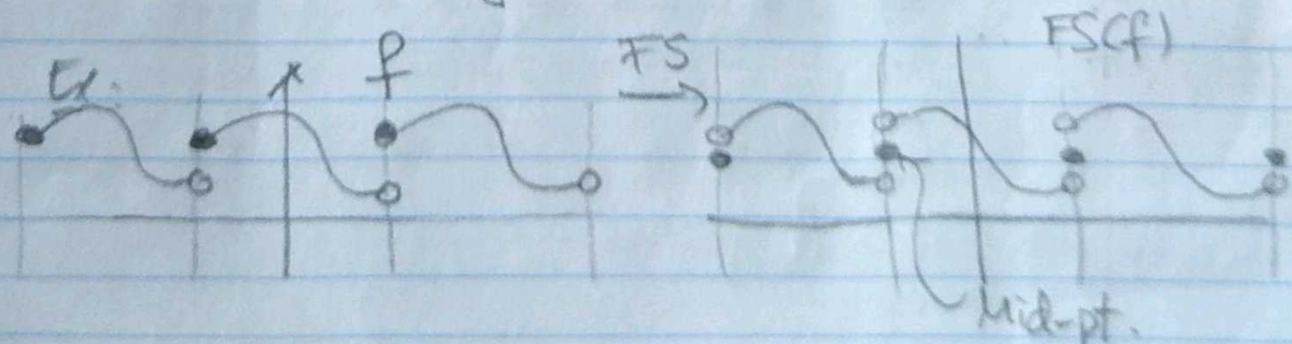
Extensions:

* Domain $[-L, L] \longrightarrow [0, L] \rightarrow \int_0^L$
  $\rightarrow$ one needs adjusting the $\int S_n S_m$, $\int C_n C_m$
* Any domain $[x_1, x_2] \rightarrow \int_{x_1}^{x_2}$
  $\rightarrow$ arguments $\cos \frac{n\pi}{L}(x - x_1)$, $\sin \frac{n\pi}{L}(x - x_1)$
  $\hookrightarrow$ same formulas with $2L \rightarrow x_2 - x_1$

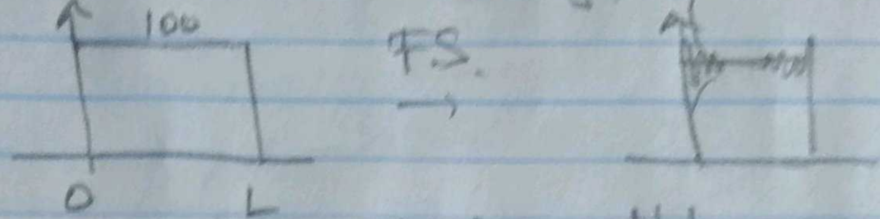
Symmetries/Parity (on $[-L, L]$)

* f odd $\rightarrow$ only Sine-Fouris series
* f even $\rightarrow$ only cosine-FS.

Properties: where f is discontinuous (inc. the boundaries), FS converges to MID-POINT.



Gibbs effect: @ discontinuities, FS does not do that well and artificial oscillations are seeing:



→ see matlab example

term-by-term $\begin{cases} \nearrow \text{differentiation} \\ \searrow \text{integration} \end{cases}$

If f continuous, the one can do term-by-term d/dx or $\int dx$.

$$f(x) = a_0 + \Sigma a_n C_n + \Sigma b_n S_n$$

$$\Rightarrow \quad f'(x) = -\sum_{n=1}^{\infty} a_n \frac{n\pi}{L} \sin \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \frac{n\pi}{L} \cos \frac{n\pi}{L} x$$

and: $$\int f(x)\,dx = a_0 x + \sum_{n=1}^{\infty} a_n \frac{L}{n\pi} \sin \frac{n\pi}{L} x - \sum_{n=1}^{\infty} b_n \frac{L}{n\pi} \cos \frac{n\pi}{L} x + C_1$$

Complex Fourier-Series

* use the fact that $e^{i\theta} = \cos\theta + i \sin\theta$

↳ repeat above & use ⊥ sin & cos

$\mathbb{R}$ (but it could be $\mathbb{C}$)

$$\Rightarrow \boxed{\begin{gathered} f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-\frac{in\pi}{L}x}, \quad c_n \in \mathbb{C} \\ c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{+\frac{in\pi}{L}x}\, dx \end{gathered}}$$

Complex F-S.

⚠ when dealing with complex, the projection (dot product) includes a complex conjugation:

f & g $\in \mathbb{C}$:

$$\langle f(x), g(x)\rangle = \int \underbrace{\overline{f(x)}}_{c.c.}\, g(x)\, dx$$

If f & g $\in \mathbb{R}$ $\Rightarrow$ $\langle f, g\rangle = \int f g\, dx$

$\langle \hat{i}, \hat{i}\rangle = \|\hat{i}\|^2$ $\quad$ $\langle f, g\rangle = \int \bar{f} f$

$= \int |f|^2$