4.8 EIGENVALUES OF 2 × 2 MATRICES

We now discuss how to find eigenvalues of 2×2 matrices in a way that does not depend explicitly on finding eigenvectors. This direct method will show that eigenvalues can be complex as well as real.

We begin the discussion with a general square matrix. Let A be an $n \times n$ matrix. Recall that $\lambda \in \mathbf{R}$ is an eigenvalue of A if there is a nonzero vector $v \in \mathbf{R}^n$ for which

$$Av = \lambda v. \tag{4.8.1}$$

The vector v is called an *eigenvector*. We may rewrite (4.8.1) as

$$(A - \lambda I_n)v = 0.$$

Since v is nonzero, it follows that if λ is an eigenvalue of A, then the matrix $A - \lambda I_n$ is singular.

Conversely, suppose that $A - \lambda I_n$ is singular for some real number λ. Then Theorem 3.7.8 implies that there is a nonzero vector $v \in \mathbf{R}^n$ such that $(A - \lambda I_n)v = 0$. Hence (4.8.1) holds and λ is an eigenvalue of A. So, if we had a direct method for determining when a matrix is singular, then we would have a method for determining eigenvalues.

Characteristic Polynomials

Corollary 3.8.3 states that 2×2 matrices are singular precisely when their determinant is 0. It follows that $\lambda \in \mathbf{R}$ is an eigenvalue for the 2×2 matrix A precisely when

$$\det(A - \lambda I_2) = 0. \tag{4.8.2}$$

We can compute (4.8.2) explicitly as follows: Note that

$$A - \lambda I_2 = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}.$$

Therefore

$$\begin{aligned} \det(A - \lambda I_2) &= (a - \lambda)(d - \lambda) - bc \\ &= \lambda^2 - (a + d)\lambda + (ad - bc). \end{aligned} \tag{4.8.3}$$

Definition 4.8.1 *The* characteristic polynomial *of the matrix A is*

$$p_A(\lambda) = \det(A - \lambda I_2).$$

For an $n \times n$ matrix $A = (a_{ij})$, we define the *trace* of A to be the sum of the diagonal elements of A; that is

$$\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}. \tag{4.8.4}$$

Thus, using (4.8.3), we can rewrite the characteristic polynomial for 2×2 matrices as

$$p_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A). \qquad \textbf{(4.8.5)}$$

As an example, consider the 2×2 matrix

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}. \qquad \textbf{(4.8.6)}$$

Then

$$A - \lambda I_2 = \begin{pmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{pmatrix}$$

and

$$p_A(\lambda) = (2-\lambda)(4-\lambda) - 3 = \lambda^2 - 6\lambda + 5.$$

It is now easy to verify (4.8.5) for (4.8.6).

Eigenvalues

For 2×2 matrices A, $p_A(\lambda)$ is a quadratic polynomial. As we have discussed, the real roots of p_A are real eigenvalues of A. For 2×2 matrices we now generalize our first definition of eigenvalues, Definition 4.7.1, to include complex eigenvalues.

Definition 4.8.2 *An* eigenvalue *of A is a root of the characteristic polynomial p_A.*

Suppose that λ_1 and λ_2 are the roots of p_A. It follows that

$$p_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2. \qquad \textbf{(4.8.7)}$$

Equating the two forms of p_A (4.8.5) and (4.8.7) shows that

$$\mathrm{tr}(A) = \lambda_1 + \lambda_2 \qquad \textbf{(4.8.8)}$$

$$\det(A) = \lambda_1\lambda_2. \qquad \textbf{(4.8.9)}$$

Thus, for 2×2 matrices, the trace is the sum of the eigenvalues and the determinant is the product of the eigenvalues. In Theorems 8.2.4(b) and 8.2.9 we show that these statements are also valid for $n \times n$ matrices.

Recall that in example (4.8.6) the characteristic polynomial is

$$p_A(\lambda) = \lambda^2 - 6\lambda + 5 = (\lambda - 5)(\lambda - 1).$$

Thus the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 5$, and identities (4.8.8) and (4.8.9) are easily verified for this example.

Next we consider an example with complex eigenvalues and verify that these identities are equally valid in this instance. Let

$$B = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}.$$

The characteristic polynomial is

$$p_B(\lambda) = \lambda^2 - 6\lambda + 11.$$

Using the quadratic formula, we see that the roots of p_B (that is, the eigenvalues of B) are

$$\lambda_1 = 3 + i\sqrt{2} \quad \text{and} \quad \lambda_2 = 3 - i\sqrt{2}.$$

Again the sum of the eigenvalues is 6, which equals the trace of B, and the product of the eigenvalues is 11, which equals the determinant of B.

Since the characteristic polynomial of 2×2 matrices is always a quadratic polynomial, it follows that 2×2 matrices have precisely two eigenvalues—including multiplicity. Properties of these eigenvalues are described as follows. The *discriminant* of A is

$$D = [\mathrm{tr}(A)]^2 - 4\det(A). \tag{4.8.10}$$

Theorem 4.8.3 *There are three possibilities for the two eigenvalues of a 2×2 matrix A that we can describe in terms of the discriminant:*

(a) The eigenvalues of A are real and distinct ($D > 0$).

(b) The eigenvalues of A are a complex conjugate pair ($D < 0$).

(c) The eigenvalues of A are real and equal ($D = 0$).

Proof: We can find the roots of the characteristic polynomial using the form of p_A given in (4.8.5) and the quadratic formula. The roots are

$$\frac{1}{2}\left(\mathrm{tr}(A) \pm \sqrt{[\mathrm{tr}(A)]^2 - 4\det(A)}\right) = \frac{\mathrm{tr}(A) \pm \sqrt{D}}{2}.$$

The proof of the theorem now follows. If $D > 0$, then the eigenvalues of A are real and distinct; if $D < 0$, then the eigenvalues are complex conjugates; and if $D = 0$, then the eigenvalues are real and equal. ◆

Eigenvectors

The following lemma contains an important observation about eigenvectors:

Lemma 4.8.4 *Every eigenvalue λ of a 2×2 matrix A has an eigenvector v. That is, there is a nonzero vector $v \in \mathbf{C}^2$ satisfying*

$$Av = \lambda v.$$

Proof: When the eigenvalue λ is real, we know that an eigenvector $v \in \mathbf{R}^2$ exists. However, when λ is complex, we must show that there is a complex eigenvector $v \in \mathbf{C}^2$, and this we have not yet done. More precisely, we must show that if λ is a complex root of the characteristic polynomial p_A, then there is a complex vector v such that

$$(A - \lambda I_2)v = 0.$$

As we discussed in Section 2.5, finding v is equivalent to showing that the complex matrix

$$A - \lambda I_2 = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

is not row equivalent to the identity matrix; see Theorem 2.5.2. Since a is real and λ is not, $a - \lambda \neq 0$. A short calculation shows that $A - \lambda I_2$ is row equivalent to the matrix

$$\begin{pmatrix} 1 & \dfrac{b}{a - \lambda} \\ 0 & \dfrac{p_A(\lambda)}{a - \lambda} \end{pmatrix}.$$

This matrix is not row equivalent to the identity matrix, since $p_A(\lambda) = 0$. ◆

An Example of a Matrix with Real Eigenvectors

Once we know the eigenvalues of a 2×2 matrix, the associated eigenvectors can be found by direct calculation. For example, we showed previously that the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

in (4.8.6) has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 5$. With this information we can find the associated eigenvectors. To find an eigenvector associated with the eigenvalue $\lambda_1 = 1$ compute

$$A - \lambda_1 I_2 = A - I_2 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}.$$

It follows that $v_1 = (3, -1)^t$ is an eigenvector, since

$$(A - I_2)v_1 = 0.$$

Similarly, to find an eigenvector associated with the eigenvalue $\lambda_2 = 5$ compute

$$A - \lambda_2 I_2 = A - 5I_2 = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}.$$

It follows that $v_2 = (1, 1)^t$ is an eigenvector, since

$$(A - 5I_2)v_2 = 0.$$

Examples of Matrices with Complex Eigenvectors

Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $p_A(\lambda) = \lambda^2 + 1$ and the eigenvalues of A are $\pm i$. To find the eigenvector $v \in \mathbf{C}^2$ whose existence is guaranteed by Lemma 4.8.4, we need to solve the complex system of linear equations $Av = iv$. We can rewrite this system as

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

A calculation shows that

$$v = \begin{pmatrix} i \\ 1 \end{pmatrix} \tag{4.8.11}$$

is a solution. Since the coefficients of A are real, we can take the complex conjugate of the equation $Av = iv$ to obtain

$$A\overline{v} = -i\overline{v}.$$

Thus

$$\overline{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

is the eigenvector corresponding to the eigenvalue $-i$. This comment is valid for any complex eigenvalue.

More generally, let

$$A = \begin{pmatrix} \sigma & -\tau \\ \tau & \sigma \end{pmatrix}, \tag{4.8.12}$$

where $\tau \neq 0$. Then

$$\begin{aligned} p_A(\lambda) &= \lambda^2 - 2\sigma\lambda + \sigma^2 + \tau^2 \\ &= (\lambda - (\sigma + i\tau))(\lambda - (\sigma - i\tau)), \end{aligned}$$

and the eigenvalues of A are the complex conjugates $\sigma \pm i\tau$. Thus A has no real eigenvectors. The complex eigenvectors of A are v and $\overline{v}$, where v is defined in (4.8.11).

HAND EXERCISES

1. For which values of λ is the matrix

$$\begin{pmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{pmatrix}$$

not invertible? **Note:** These values of λ are just the eigenvalues of the matrix $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

In Exercises 2–5, compute the determinant, trace, and characteristic polynomials for the given 2×2 matrix.

2. $\begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}$ **3.** $\begin{pmatrix} 2 & 13 \\ -1 & 5 \end{pmatrix}$ **4.** $\begin{pmatrix} 1 & 4 \\ 1 & -1 \end{pmatrix}$ **5.** $\begin{pmatrix} 4 & 10 \\ 2 & 5 \end{pmatrix}$

In Exercises 6–8, compute the eigenvalues for each 2×2 matrix.

6. $\begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix}$ **7.** $\begin{pmatrix} -3 & 2 \\ 1 & 0 \end{pmatrix}$ **8.** $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$

9.

(a) Let A and B be 2×2 matrices. Using direct calculation, show that

$$\text{tr}(AB) = \text{tr}(BA). \qquad \textbf{(4.8.13)}$$

(b) Now let A and B be $n \times n$ matrices. Verify by direct calculation that (4.8.13) is still valid.

COMPUTER EXERCISES

In Exercises 10–12, use the program `map` to guess whether the given matrix has real or complex conjugate eigenvalues. For each example, write the reasons for your guess.

10. $A = \begin{pmatrix} 0.97 & -0.22 \\ 0.22 & 0.97 \end{pmatrix}$ **11.** $B = \begin{pmatrix} 0.97 & 0.22 \\ 0.22 & 0.97 \end{pmatrix}$ **12.** $C = \begin{pmatrix} 0.4 & -1.4 \\ 1.5 & 0.5 \end{pmatrix}$

In Exercises 13–14, use the program `map` to guess one of the eigenvectors of the given matrix. What is the corresponding eigenvalue? Using `map`, can you find a second eigenvalue and eigenvector?

13. $A = \begin{pmatrix} 2 & 4 \\ 2 & 0 \end{pmatrix}$ **14.** $B = \begin{pmatrix} 2 & -1 \\ 0.25 & 1 \end{pmatrix}$

Hint Use the feature `rescale` in the `MAP Options`. Then the length of the vector is rescaled to 1 after each use of the command `map`. In this way you can avoid overflows in the computations while still being able to see the directions where the vectors are moved by the matrix mapping.

15. The MATLAB command `eig` computes the eigenvalues of matrices. Use `eig` to compute the eigenvalues of $A = \begin{pmatrix} 2.34 & -1.43 \\ \pi & e \end{pmatrix}$.