(a) For what values of λ is $\det(\lambda A - B) = 0$?
(b) Is there a vector x for which $Ax = Bx$?

In Exercises 10 and 11, verify that the given matrix has determinant -1.

10. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ **11.** $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

12. Compute the cofactor matrices A_{13}, A_{22}, and A_{21} when $A = \begin{pmatrix} 3 & 2 & -4 \\ 0 & 1 & 5 \\ 0 & 0 & 6 \end{pmatrix}$.

13. Compute the cofactor matrices B_{11}, B_{23}, and B_{43} when $B = \begin{pmatrix} 0 & 2 & -4 & 5 \\ -1 & 7 & -2 & 10 \\ 0 & 0 & 0 & -1 \\ 3 & 4 & 2 & -10 \end{pmatrix}$.

14. Find values of λ where the determinant of the matrix

$$A_\lambda = \begin{pmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 1 & 1 \\ -1 & 1 & \lambda \end{pmatrix}$$

vanishes.

15. Suppose that two $n \times p$ matrices A and B are row equivalent. Show that there is an invertible $n \times n$ matrix P such that $B = PA$.

16. Let A be an invertible $n \times n$ matrix and let $b \in \mathbf{R}^n$ be a column vector. Let B_j be the $n \times n$ matrix obtained from A by replacing the jth column of A by the vector b. Let $x = (x_1, \ldots, x_n)^t$ be the unique solution to $Ax = b$. Then Cramer's rule states that

$$x_j = \frac{\det(B_j)}{\det(A)}. \qquad \textbf{(8.1.13)}$$

Prove Cramer's rule.

Hint Let A_j be the jth column of A so that $A_j = Ae_j$. Show that

$$B_j = A(e_1|\cdots|e_{j-1}|x|e_{j+1}|\cdots|e_n).$$

Using this product, compute the determinant of B_j and verify (8.1.13).

8.2 EIGENVALUES

In this section we discuss how to find eigenvalues for an $n \times n$ matrix A. This discussion parallels the discussion for 2×2 matrices given in Section 4.8. As we noted in that section, λ is a real eigenvalue of A if there exists a nonzero eigenvector v such that

$$Av = \lambda v. \qquad \textbf{(8.2.1)}$$

It follows that the matrix $A - \lambda I_n$ is singular, since

$$(A - \lambda I_n)v = 0.$$

Theorem 8.1.7 implies that

$$\det(A - \lambda I_n) = 0.$$

With these observations in mind, we can make the following definition:

Definition 8.2.1 *Let A be an $n \times n$ matrix. The* characteristic polynomial *of A is*

$$p_A(\lambda) = \det(A - \lambda I_n).$$

In Theorem 8.2.3 we show that $p_A(\lambda)$ is indeed a polynomial of degree n in λ. Note here that the roots of p_A are the *eigenvalues* of A. As we discussed, the real eigenvalues of A are roots of the characteristic polynomial. Conversely, if λ is a real root of p_A, then Theorem 8.1.7 states that the matrix $A - \lambda I_n$ is singular and therefore that there exists a nonzero vector v such that (8.2.1) is satisfied. Similarly, by using this extended algebraic definition of eigenvalues, we allow the possibility of complex eigenvalues. The complex analog of Theorem 8.1.7 shows that if λ is a complex eigenvalue, then there exists a nonzero complex n-vector v such that (8.2.1) is satisfied.

Example 8.2.2 *Let A be an $n \times n$ lower triangular matrix. Then the diagonal entries are the eigenvalues of A.*

We verify this statement as follows:

$$A - \lambda I_n = \begin{pmatrix} a_{11} - \lambda & & 0 \\ & \ddots & \\ (*) & & a_{nn} - \lambda \end{pmatrix}.$$

Since the determinant of a triangular matrix is the product of the diagonal entries, it follows that

$$p_A(\lambda) = (a_{11} - \lambda) \cdots (a_{nn} - \lambda) \tag{8.2.2}$$

and hence that the diagonal entries of A are roots of the characteristic polynomial. A similar argument works if A is upper triangular.

It follows from (8.2.2) that the characteristic polynomial of a triangular matrix is a polynomial of degree n and that

$$p_A(\lambda) = (-1)^n \lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_0 \tag{8.2.3}$$

for some real constants $b_0, \ldots, b_{n-1}$. In fact, this statement is true in general.

Theorem 8.2.3 *Let A be an $n \times n$ matrix. Then p_A is a polynomial of degree n of the form (8.2.3).*

Proof: Let C be an $n \times n$ matrix whose entries have the form $c_{ij} + d_{ij}\lambda$. Then $\det(C)$ is a polynomial in λ of degree at most n. We verify this statement by induction. It is easily verified when $n = 1$, since then $C = (c + d\lambda)$ for some real numbers c and d. Then $\det(C) = c + d\lambda$, which is a polynomial of degree at most one. (It may have degree zero if $d = 0$.) So assume that this statement is true for $(n-1) \times (n-1)$ matrices. Recall from (8.1.9) that

$$\det(C) = (c_{11} + d_{11}\lambda)\det(C_{11}) + \cdots + (-1)^{n+1}(c_{1n} + d_{1n}\lambda)\det(C_{1n}).$$

By induction each of the determinants C_{1j} is a polynomial of degree at most $n-1$. It follows that multiplication by $c_{1j} + d_{1j}\lambda$ yields a polynomial of degree at most n in λ. Since the sum of polynomials of degree at most n is a polynomial of degree at most n, we have verified our assertion.

Since $A - \lambda I_n$ is a matrix whose entries have the desired form, it follows that $p_A(\lambda)$ is a polynomial of degree at most n in λ. To complete the proof of this theorem we need to show that the coefficient of λ^n is $(-1)^n$. Again we verify this statement by induction. This statement is easily verified for 1×1 matrices; we assume that it is true for $(n-1) \times (n-1)$ matrices. Again use (8.1.9) to compute

$$\det(A - \lambda I_n) = (a_{11} - \lambda)\det(B_{11}) - a_{12}\det(B_{12}) + \cdots + (-1)^{n+1}a_{1n}\det(B_{1n}),$$

where B_{1j} are the cofactor matrices of $A - \lambda I_n$. From our previous observation, all of the terms $\det(B_{1j})$ are polynomials of degree at most $n-1$. Thus, in this expansion, the only term that can contribute a term of degree n is

$$-\lambda\det(B_{11}).$$

Note that the cofactor matrix B_{11} is the $(n-1) \times (n-1)$ matrix

$$B_{11} = A_{11} - \lambda I_{n-1},$$

where A_{11} is the first cofactor matrix of the matrix A. By induction, $\det(B_{11})$ is a polynomial of degree $n-1$ with leading term $(-1)^{n-1}\lambda^{n-1}$. Multiplying this polynomial by $-\lambda$ yields a polynomial of degree n with the correct leading term. ◆

General Properties of Eigenvalues

The *fundamental theorem of algebra* states that every polynomial of degree n has exactly n roots (counting multiplicity). For example, the quadratic formula shows that every quadratic polynomial has exactly two roots. In general, the proof of the fundamental theorem is not easy and is certainly beyond the limits of this course.

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Indeed, the difficulty in proving the *fundamental theorem of algebra* is to prove that a polynomial $p(\lambda)$ of degree $n > 0$ has one (complex) root. Suppose that λ_0 is a root of $p(\lambda)$; that is, suppose that $p(\lambda_0) = 0$. Then it is easy to show that

$$p(\lambda) = (\lambda - \lambda_0)q(\lambda) \tag{8.2.4}$$

for some polynomial q of degree $n - 1$. So once we know that p has a root, we can argue by induction to prove that p has n roots.

Recall that a polynomial need not have any real roots. For example, the polynomial $p(\lambda) = \lambda^2 + 1$ has no real roots, since $p(\lambda) > 0$ for all real λ. This polynomial does have two complex roots $\pm i = \pm\sqrt{-1}$.

However, a polynomial with real coefficients has either real roots or complex roots that come in complex conjugate pairs. To verify this statement, we need to show that if λ_0 is a complex root of $p(\lambda)$, then so is $\overline{\lambda_0}$. We claim that

$$p(\overline{\lambda}) = \overline{p(\lambda)}.$$

To verify this point, suppose that

$$p(\lambda) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0,$$

where each $c_j \in \mathbf{R}$. Then

$$\overline{p(\lambda)} = \overline{c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0} = c_n\overline{\lambda}^n + c_{n-1}\overline{\lambda}^{n-1} + \cdots + c_0 = p(\overline{\lambda}).$$

If λ_0 is a root of $p(\lambda)$, then

$$p(\overline{\lambda_0}) = \overline{p(\lambda_0)} = \overline{0} = 0.$$

Hence $\overline{\lambda_0}$ is also a root of p.

It follows that:

Theorem 8.2.4 *Every (real) $n \times n$ matrix A has exactly n eigenvalues $\lambda_1, \ldots, \lambda_n$. These eigenvalues are either real or complex conjugate pairs. Moreover,*

(a) $p_A(\lambda) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$, *and*

(b) $\det(A) = \lambda_1 \cdots \lambda_n$.

Proof: Since the characteristic polynomial p_A is a polynomial of degree n with real coefficients, the first part of the theorem follows from the preceding discussion. In particular, it follows from (8.2.4) that

$$p_A(\lambda) = c(\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$$

for some constant c. Formula (8.2.3) implies that $c = 1$, which proves (a). Since $p_A(\lambda) = \det(A - \lambda I_n)$, it follows that $p_A(0) = \det(A)$. Thus (a) implies that $p_A(0) = \lambda_1 \cdots \lambda_n$, thus proving (b). ◆

The eigenvalues of a matrix do not have to be different. For example, consider the extreme case of a strictly triangular matrix A. Example 8.2.2 shows that all of the eigenvalues of A are 0.

We now discuss certain properties of eigenvalues.

Corollary 8.2.5 *Let A be an $n \times n$ matrix. Then A is invertible if and only if 0 is* not *an eigenvalue of A.*

Proof: The proof follows from Theorem 8.1.7 and Theorem 8.2.4(b). ◆

Lemma 8.2.6 *Let A be a singular $n \times n$ matrix. Then the null space of A is the span of all eigenvectors whose associated eigenvalue is 0.*

Proof: An eigenvector v of A has eigenvalue 0 if and only if

$$Av = 0.$$

This statement is valid if and only if v is in the null space of A. ◆

Theorem 8.2.7 *Let A be an invertible $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then the eigenvalues of A^{-1} are $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$.*

Proof: We claim that

$$p_A(\lambda) = (-1)^n \det(A)\lambda^n p_{A^{-1}}\left(\frac{1}{\lambda}\right).$$

It then follows that $1/\lambda$ is an eigenvalue for A^{-1} for each eigenvalue λ of A. This makes sense, since the eigenvalues of A are nonzero.

Compute:

$$\begin{aligned}
(-1)^n \det(A)\lambda^n p_{A^{-1}}\left(\frac{1}{\lambda}\right) &= (-\lambda)^n \det(A) \det\left(A^{-1} - \frac{1}{\lambda} I_n\right) \\
&= \det(-\lambda A) \det\left(A^{-1} - \frac{1}{\lambda} I_n\right) \\
&= \det\left(-\lambda A\left(A^{-1} - \frac{1}{\lambda} I_n\right)\right) \\
&= \det(A - \lambda I_n) \\
&= p_A(\lambda),
\end{aligned}$$

which verifies the claim. ◆

Theorem 8.2.8 *Let A and B be similar $n \times n$ matrices. Then*

$$p_A = p_B,$$

and hence the eigenvalues of A and B are identical.

Proof: Since B and A are similar, there exists an invertible $n \times n$ matrix S such that $B = S^{-1}AS$. It follows that

$$\det(B - \lambda I_n) = \det(S^{-1}AS - \lambda I_n) = \det(S^{-1}(A - \lambda I_n)S) = \det(A - \lambda I_n),$$

which verifies that $p_A = p_B$. ◆

Recall that the *trace* of an $n \times n$ matrix A is the sum of the diagonal entries of A; that is,

$$\text{tr}(A) = a_{11} + \cdots + a_{nn}.$$

We state without proof the following theorem:

Theorem 8.2.9 *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then*

$$tr(A) = \lambda_1 + \cdots + \lambda_n.$$

It follows from Theorem 8.2.8 that the traces of similar matrices are equal.

MATLAB Calculations

The commands for computing characteristic polynomials and eigenvalues of square matrices are straightforward in MATLAB. In particular, for an $n \times n$ matrix A, the MATLAB command `poly(A)` returns the coefficients of $(-1)^n p_A(\lambda)$.

For example, reload the 4×4 matrix A of (8.1.12) by typing e8_1_11. The characteristic polynomial of A is found by typing

```
poly(A)
```

to obtain

```
ans =
    1.0000   -5.0000   15.0000  -10.0000  -46.0000
```

Thus the characteristic polynomial of A is

$$p_A(\lambda) = \lambda^4 - 5\lambda^3 + 15\lambda^2 - 10\lambda - 46.$$

The eigenvalues of A are found by typing `eig(A)` and obtaining

```
ans =
  -1.2224
   1.6605 + 3.1958i
   1.6605 - 3.1958i
   2.9014
```

Thus A has two real eigenvalues and one complex conjugate pair of eigenvalues. Note that MATLAB has preprogrammed not only the algorithm for finding the characteristic polynomial, but also numerical routines for finding the roots of the characteristic polynomial.

The trace of A is found by typing `trace(A)` and obtaining

```
ans =
     5
```

Using the MATLAB command `sum`, we can verify the statement of Theorem 8.2.9. Indeed `sum(v)` computes the sum of the components of the vector v, and typing

```
sum(eig(A))
```

we obtain the answer `5.0000`, as expected.

HAND EXERCISES

In Exercises 1 and 2, determine the characteristic polynomial and the eigenvalues of each matrix.

1. $A = \begin{pmatrix} -9 & -2 & -10 \\ 3 & 2 & 3 \\ 8 & 2 & 9 \end{pmatrix}$

2. $B = \begin{pmatrix} 2 & 1 & -5 & 2 \\ 1 & 2 & 13 & 2 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

3. Find a basis for the eigenspace of

$$A = \begin{pmatrix} 3 & 1 & -1 \\ -1 & 1 & 1 \\ 2 & 2 & 0 \end{pmatrix}$$

corresponding to the eigenvalue $\lambda = 2$.

4. Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

(a) Verify that the characteristic polynomial of A is $p_\lambda(A) = (\lambda - 1)(\lambda + 2)^2$.
(b) Show that $(1, 1, 1)$ is an eigenvector of A corresponding to $\lambda = 1$.
(c) Show that $(1, 1, 1)$ is orthogonal to every eigenvector of A corresponding to the eigenvalue $\lambda = -2$.

5. Consider the matrix $A = \begin{pmatrix} 8 & 5 \\ -10 & -7 \end{pmatrix}$.
(a) Find the eigenvalues and eigenvectors of A.
(b) Show that the eigenvectors found in (a) form a basis for $\mathbf{R}^2$.
(c) Find the coordinates of the vector (x_1, x_2) relative to the basis in (b).

6. Find the characteristic polynomial and the eigenvalues of

$$A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix}.$$

Find eigenvectors corresponding to each of the three eigenvalues.

7. Let A be an $n \times n$ matrix. Suppose that

$$A^2 + A + I_n = 0.$$

Prove that A is invertible.

In Exercises 8 and 9, decide whether the given statements are *true* or *false*. If a statement is false, give a counterexample; if a statement is true, give a proof.

8. If the eigenvalues of a 2×2 matrix are equal to 1, then the four entries of that matrix are each less than 500.

9. The trace of the product of two $n \times n$ matrices is the product of the traces.

10. When n is odd, show that every real $n \times n$ matrix has a real eigenvalue.

COMPUTER EXERCISES

In Exercises 11 and 12, (a) use MATLAB to compute the eigenvalues, traces, and characteristic polynomials of the given matrix. (b) Use the results from (a) to confirm Theorems 8.2.7 and 8.2.9.

11.

$$A = \begin{pmatrix} -12 & -19 & -3 & 14 & 0 \\ -12 & 10 & 14 & -19 & 8 \\ 4 & -2 & 1 & 7 & -3 \\ -9 & 17 & -12 & -5 & -8 \\ -12 & -1 & 7 & 13 & -12 \end{pmatrix} \quad \textbf{(8.2.5)*}$$

12.

$$B = \begin{pmatrix} -12 & -5 & 13 & -6 & -5 & 12 \\ 7 & 14 & 6 & 1 & 8 & 18 \\ -8 & 14 & 13 & 9 & 2 & 1 \\ 2 & 4 & 6 & -8 & -2 & 15 \\ -14 & 0 & -6 & 14 & 8 & -13 \\ 8 & 16 & -8 & 3 & 5 & 19 \end{pmatrix} \quad \textbf{(8.2.6)*}$$

13. Use MATLAB to compute the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 4 & -6 & 7 \\ 2 & 0 & 5 \\ -10 & 2 & 5 \end{pmatrix}.$$

Denote this polynomial by $p_A(\lambda) = -(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)$. Then compute the matrix

$$B = -(A^3 + p_2A^2 + p_1A + p_0I).$$

What do you observe? In symbols, $B = p_A(A)$. Compute the matrix B for examples of other square matrices A and determine whether or not your observation was an accident.