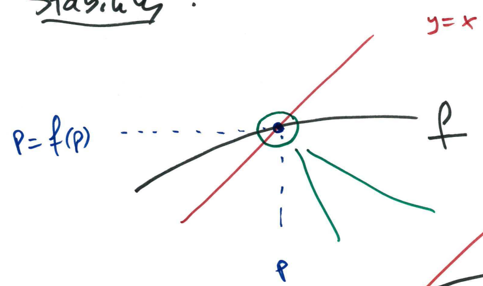
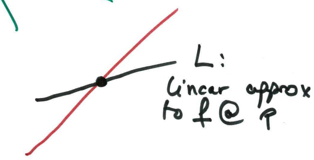
3.1

Stability:





Map: • $x_{n+1} = f(x_n)$

• $\boxed{x_n = p + \varepsilon_n}$ ← dist. from f. pt @ time = n

3.2

Iterate: $x_{n+1} = f(x_n)$

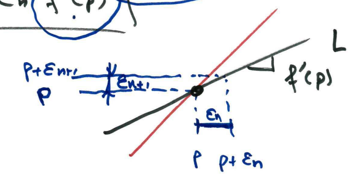
$= f(p+\varepsilon_n)$ $|\varepsilon_n| \ll 1$

$\simeq f(p) + \varepsilon_n f'(p) + \cancel{\frac{\varepsilon_n^2}{2} f''(p) + \dots} \to 0$

$\approx f(p) + \varepsilon_n f'(p)$

$\boxed{x_{n+1} \approx p + \underbrace{\varepsilon_n f'(p)}_{\varepsilon_{n+1}}}$

$\boxed{\varepsilon_{n+1} = \varepsilon_n f'(p)}$ → Contract $|f'(p)| < 1$



3.3

Def: Epsilon neighborhood:

$N_\varepsilon(p) = \{$ set of pts within a dist. ε of $p \}$

Def 1.4: let f be a map on $\mathbb{R}$ and let p be a f. pt. $[\, p = f(p) \,]$

• If ALL pts sufficiently close to p are attracted to p then p is called a SINK or ATTRACTING Fx. PT. (prop: STABLE)

That is: $\boxed{\text{if } \exists\, \varepsilon > 0 \text{ s.t. } \forall x\ N_\varepsilon(p): \lim_{k\to\infty} f^k(x) = p \Rightarrow p \text{ is a SINK}}$

• If ALL pts suff. close to p are repelled away from p then p

3.4

is called a SOURCE or REPELLING F. PT (prop: unstable)

Theo 1.5: Let f be a map that is smooth on an open interval of a f. pt. p. then:

1- If $|f'(p)| < 1 \Rightarrow p$ is SINK (attract)

2- If $|f'(p)| > 1 \Rightarrow p$ is a source (repel)

Proof: suppose $|f'(p)| < 1$.

$\Rightarrow \lim_{x\to p} \frac{|f(x)-f(p)|}{|x-p|} = |f'(p)| = a < 1$

$\Rightarrow$ there exist an $N_\varepsilon(p)$ such that $\frac{|f(x)-f(p)|}{|x-p|} < a$

$\Rightarrow |f(x) - f(p)| < a|x-p|$

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$\Rightarrow \underbrace{|f(x) - p|}_{\text{dist. after}} < \underset{a<1}{a}\ \underbrace{|x-p|}_{\text{dist. before}}$

$\Rightarrow$ I'm getting closer

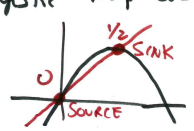
... $|f^k(x) - p| < a^k |x-p|$

$a < 1 \Rightarrow a^k \to 0$

$\Rightarrow$ if $k \to \infty$ $a^k \to 0 \Rightarrow |f^k(x) - p| \to 0$

$\Rightarrow \boxed{f^k(x) \xrightarrow{k\to\infty} p}$ ■

Ex: logistic map $a = 2$: $g_2(x) = 2x(1-x)$



$\left.\begin{array}{l} x_1^* = 0 \\ x_2^* = 1/2 \end{array}\right\} g_2(x) = x$

3.6

Stab: * $x_1^* = 0$ $[\, g_2(x) = 2x(1-x) = 2x - 2x^2$

$[\, g_2'(x) = 2 - 4x = 2(1-2x)$

$f'(x_1^*) = 2(1 - 2\cdot 0) = 2(1) = 2 \Rightarrow |f'(x_1^*)| > 1$

$\Rightarrow \boxed{x_1^* \text{ SOURCE}}$

* $x_2^* = 1/2 \Rightarrow f'(x_2^*) = 2(1 - 2\cdot 1/2) = 2(1-1) = 0$

$\Rightarrow f'(x_2^*) = 0 \Rightarrow |f'(x_2^*)| < 1$

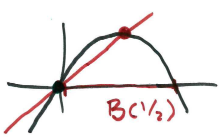
$\Rightarrow \boxed{x_2^* \text{ SINK}}$

Def: If $f'(p) = 0 \Rightarrow p$ is SUPERSTABLE

Def: the set of pts that converge towards an attracting f. pt is called the BASIN of attraction of the f. pt p. Denote by $B(p)$

3.7

Ex: Basins of $g_2(x) = 2x(1-x)$



$B(1/2) = (0,1)$

$B(0) = \{0, 1\}$

$B(-\infty) = (-\infty, 0) \cup (1, +\infty)$

Proof: • If we are in $B(1/2) \Rightarrow |x_{n+1} - 1/2| < |x_n - 1/2|$

• when is this ineq true:

$|x_{n+1} - 1/2| < |x_n - 1/2|$

$\Rightarrow |g_2(x_n) - 1/2| < |x_n - 1/2|$

$\Rightarrow |2x_n(1-x_n) - 1/2| < |x_n - 1/2|$

$\Rightarrow 2|x_n - x_n^2 - 1/4| < |x_n - 1/2|$

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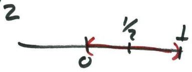
$\Rightarrow 2|-(x_n - 1/2)^2| < |x_n - 1/2|$

$\Rightarrow 2|x_n - 1/2|\,\cancel{|x_n - 1/2|} < \cancel{|x_n - 1/2|}$

$\Rightarrow |x_n - 1/2| < 1/2$

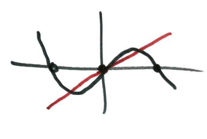


$\Rightarrow \boxed{x_n \in (0,1)}$

$\Rightarrow B(1/2) = (0,1)$

Ex: consider $f(x) = \frac{3x - x^3}{2}$

$= x\frac{(3-x^2)}{2}$



F. pts: $x = f(x)$

$x = \cancel{x}\frac{(3-x^2)}{2}$ $\boxed{x_1^* = 0}$

$2 = 3 - x^2 \Rightarrow x^2 = 1$

$\Rightarrow x_2^* = -1,$

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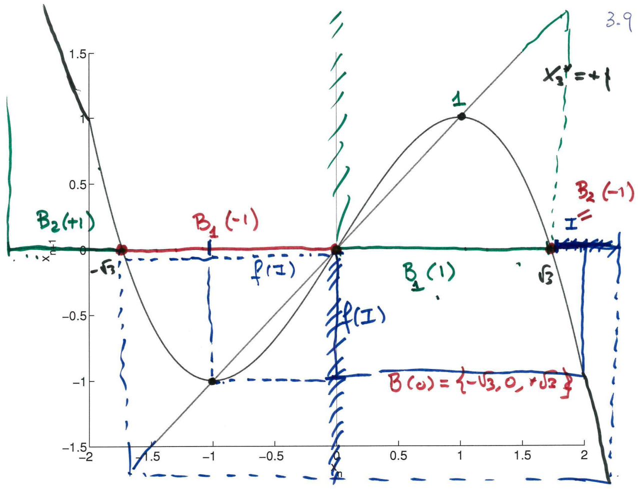
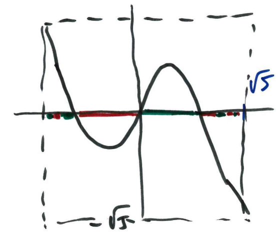


3.10



$\boxed{\sqrt{5}}$: $\left.\begin{array}{l} f(\sqrt{5}) = -\sqrt{5} \\ f(-\sqrt{5}) = \sqrt{5} \end{array}\right\}$ $f(f(\sqrt{5})) = \sqrt{5}$

$f^2(\sqrt{5}) = \sqrt{5}$

$h = f^2 \Rightarrow h(\sqrt{5}) = \sqrt{5}$

3.11

Sec 1.4 Periodic orbits

logistic map: $g_a(x) = a\,x(1-x)$

* $\underline{a=2}$: $x_1^* = 0$, $x_2^* = 1/2$ f. pt.
* $\underline{a=3.3}$: Stab: $|f'(x_1^*)| > 1$, $|f'(x_2^*)| = 0 < 1$

In gral. $\forall a$: $g_a(x) = x$

$\Rightarrow ax(1-x) = x \Rightarrow \boxed{x_1^* = 0}$

$\Rightarrow a(1-x) = 1$

$\Rightarrow (1-x) = 1/a = \boxed{x_2^* = 1 - 1/a}$
$= \frac{a-1}{a}$

* $a = 3.3 \Rightarrow x_1^* = 0$

$x_2^* = 1 - 1/3.3 = 1 - \frac{10}{33} = \frac{33-10}{33} = \frac{23}{33}$

$g_a(x) = ax(1-x) = ax - ax^2 \Rightarrow g_a' = a - 2ax = a(1-2x)$

3.12

• $f'(x_1^*) = f'(0) = \underline{a}$

If $\begin{cases} |a| < 1 \Rightarrow x_1^* = 0 \text{ is } S \\ |a| > 1 \Rightarrow x_1^* = 0 \text{ is } U \end{cases}$

• $f'(x_2^*) = a(1 - 2x_2^*) =$

$= a\left(1 - 2\frac{(a-1)}{a}\right) = a - 2(a-1)$

$= a - 2a + 2 = \underline{\underline{2-a}}$

* $a = 3.3$: ∘ $f'(x_1^*) = a = 3.3 > 1 \Rightarrow U$

∘ $f'(x_2^*) = 2 - 3.3 = -1.3$

$|f'(x_2^*)| > 1 \Rightarrow U$

? where do orbits go if No f pt is S???