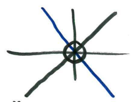
4.1

**Def 1.6:** • let $f$ be a map. A periodic pt. of period $k$ is a pt $p$ such that $f^k(p) = p$ and $k$ is the smallest possible integer.

• The orbit emanating from $p$ is called a periodic orbit of period $k$ $\equiv$ period-$k$ orbit.

**Ex:** $f(x) = -x$



• fixed pt: $x^* = 0$

• every pt except $x^* = 0$ is a period-2 pt since $f^2(x) = -(-x) = x$

**Stability:** period-$k$ pt $\longrightarrow$ stability for $f^k(p)$

4.2

**Def: 1.8:** A period-$k$ pt $p$ is
* attractive / sink iff. $p$ is a sink of $f^k$
* repulsive / source iff. $p$ is a source of $f^k$

**Ex:** • particular case with $k=2$

• $p_1$ is a period-2 pt
$\longrightarrow$ generates orbit $\{p_1, p_2, p_1, p_2, \dots\}$
orbit $= \{p_1, p_2\}$
and $p_2 \neq p_1$

• $p_2 = f(p_1) \Rightarrow p_1 = f(p_2) = f(f(p_1))$
$= f^2(p_1)$

• def. the SECOND iterate of the map:
$$h(x) \equiv f^2(x)$$
$p_1 = h(p_1)$ & $p_2 = h(p_2)$

4.3

• Stab: $|h'(p_1)| < 1 \Rightarrow p_1$ is stab.
equiv(?) $|h'(p_2)| < 1$

• ~~Break~~ expand it:

× $h'(p_1) = [f^2(p_1)]' = [f(f(p_1))]'$
$= [f(p_1)]' \cdot f'(f(p_1))$
$= f'(p_1) \cdot f'(p_2)$

× $h'(p_2) = [f^2(p_2)]' = [f(f(p_2))]'$
$= f'(p_2) \cdot f'(f(p_2))$
$= f'(p_2) \cdot f'(p_1)$

$=$

• $p_1$ & $p_2$ $\rightarrow$ same $\Rightarrow$ order does not matter.

4.4

**Ex:** • $g_{3.3}(x) = 3.3x(1-x)$
$\{p_1, p_2\} = \{0.4794, 0.8236\}$
$h'(p_{1/2}) = -0.2904$
$\Rightarrow |h'(p_1)| < 1 \Rightarrow S$

• $g_{3.5}(x) = 3.5x(1-x)$
$\{p_1, p_2\} = \{3/7, 6/7\}$
$h'(p_{1/2}) = -5/4 \Rightarrow |h'(p_1)| > 1 \Rightarrow U$

⋮

as $a$ $[ax(1-x)]$ grows new period orbits appear, they become unstable and new period orbit appear

4.5

**In general:** period-$k$ orbit: $\{p_1, \dots, p_k\}$

• Stab: $h \equiv f^k$

$h'(p_1) = [f(f^{k-1}(p_1))]'$
$= [f^{k-1}(p_1)]' \cdot f'(f^{k-1}(p_1))$
$= [f^{k-1}(p_1)]' \cdot f'(p_k)$
$= [f(f^{k-2}(p_1))]' \cdot f'(p_k)$
$= [f^{k-2}(p_1)]' \cdot f'(\underbrace{f^{k-2}(p_1)}) \cdot f'(p_k)$
$= \dots \cdot f'(p_{k-1}) \cdot f'(p_k)$

$$\boxed{h'(p_1) = \prod_{i=1}^{k} f'(p_i)}$$

4.6

**1.5 the family of logistic map**

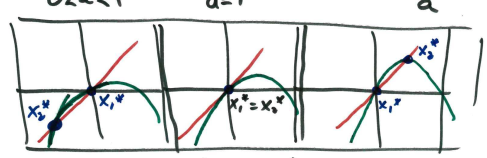
• logistic map: $g_a(x) = ax(1-x)$

• THE log. map: $g_4(x) = 4x(1-x)$

$0 < a < 4$

**f. pts:** $g_a(x) = x \Rightarrow ax(1-x) = x$ & $x_1^* = 0$
$\Rightarrow a(1-x) = 1$
$\Rightarrow x_2^* = \dfrac{a-1}{a}$

$0<a<1$ $\quad$ $a=1$




sign $(x_2^*)$: $x_2^* < 0$ if $0<a<1$
$x_2^* > 0$ if $1<a<4$

4.7

**Stab:** $g_a'(x) = (ax(1-x))' = (ax - ax^2)' = a - 2ax$
$g_a'(x) = a(1-2x)$

• $\underline{x_1^* = 0}$ : $g_a'(x_1^*) = a$ $\begin{cases} 0<a<1 & x_1^* \; S \\ 1<a<4 & x_1^* \; U \end{cases}$

• $\underline{x_2^* = 0}$ : $g_a'(x_2^*) = g_a'\left(\frac{a-1}{a}\right)$
$= a\left(1 - 2\frac{(a-1)}{a}\right)$

$= a - 2(a-1) = a - 2a + 2 = \boxed{2 - a = g_a'(x_2^*)}$

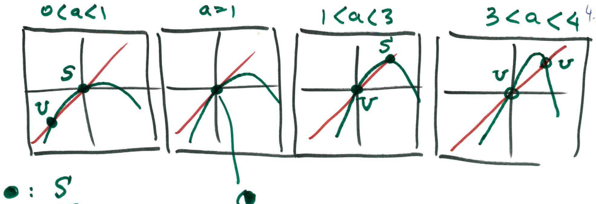
~~If $0<a<1$ $x_2^*$ $\quad 2-a<1 \Rightarrow 1<a$ $\quad 2-a>1 \Rightarrow a<1$~~

$\underline{S}$: $|g_a'(x_2^*)| < 1 \Rightarrow |2-a| < 1$
$\Rightarrow -1 < 2-a < 1 \Rightarrow a-1 < 2 < 1+a$
$\Rightarrow a < 3, \; a > 1$
$\Rightarrow 1 < a < 3$

$\underline{U}$: $a > 3$

4.8

$0<a<1$ $\quad$ $a=1$ $\quad$ $1<a<3$ $\quad$ $3<a<4$




●: S
○: U

**Period-2 orbit:** $g_a^2(x) = x$
$\Rightarrow a(\;)(1-(\;)) = x$
$g_a(x)$
$\Rightarrow a(ax(1-x))[1-(ax(1-x))] = x$
$\Rightarrow P_a^4[x] = 0$
$\Rightarrow P^2(x_1^*, x_2^*) \cdot P_a^2(x) = 0$

4.9

$$\Rightarrow (x - x_1^*) \cdot (x - x_2^*) \cdot \underbrace{P_a^2(x)}_{} = 0$$

$$\begin{cases} x_1^* = 0 & \text{Period-1} \\ x_2^* = \frac{a-1}{a} & \text{period-1} \\ x_{21/2}^* = \frac{1}{2}\left[1 + \left(\frac{1 \pm \sqrt{a^2 - 2a - 3}}{a}\right)\right] \end{cases}$$

$$\begin{cases} x_{21}^* = \\ x_{22}^* = \end{cases}$$

$$a^2 - 2a - 3 > 0 \Rightarrow \boxed{x_{21}^* \neq x_{22}^*}$$

$$g_a(x_{21}^*) = x_{22}^*$$

Stab of period-2 :

$$\left|\left[g_a^2(x_{21}^*)\right]'\right| < 1 \Rightarrow \{x_{21}^*, x_{22}^*\} \cdot S$$