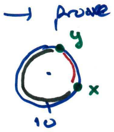
Ex T1.12 : Take $f(x) = 3x \pmod 1$

→ prove sensitive dep. to ICs

* 

$D(x,y) = \min\left(|y-x|,\ 1-|y-x|\right)$

$D(0.1, 0.4) =$  0.3

$D(0.9, 0.1) =$ 

$= 1 - |0.9 - 0.1| = 1 - 0.8 = 0.2$

* Let us take any pair of pts (x_0) & (y_0) such that $d(x_0, y_0) < 1/6$

* follow images $x_1 = f(x_0)$, $y_1 = f(y_0)$

$d(x_1, y_1)$

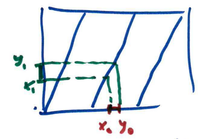
— Let us consider $d(x_1, y_1) = |y_1 - x_1|$

[the other is very similar]

$\Rightarrow d(x_1, y_1) = |y_1 - x_1| = |3y_0 \pmod 1 - 3x_0 \pmod 1|$

$= 3|(y_0 - x_0) \pmod 1|$

$d(x_1, y_1) = 3\, D(x_0, y_0)$



* Same argument n times:

$D(x_n, y_n) = 3^n D(x_0, y_0)$

→ sensitive dep. to ICs.

* Find when orbits are separated by d.

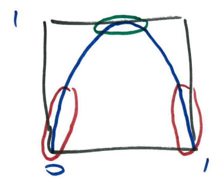
$D(x_n, y_n) = 3^n D(x_0, y_0) > d$

$\Rightarrow 3^n > \dfrac{d}{D(x_0,y_0)} \Rightarrow n \ln 3 \geq \ln\left(\dfrac{d}{d_0}\right)$

$d_0 \equiv D(x_0, y_0)$

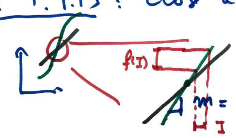
$\Rightarrow \boxed{n > \dfrac{\ln(d/d_0)}{\ln 3}} \Rightarrow D(x_n, y_n) > d$

Q: Does $G(x) = 4x(1-x)$ has sensitive dep. to ICs ?

A: 

→ ⚠ For sensitive dep. to ICs (expansion) we need "averaging"

Ex T. 1.13 : close a repeller $|f'(p)| > 1$



⟹ There is sensit. dep. to ICs close to a repeller.

And rate $= |f'(p)|$

1.8 Itineraries = Symbolic dynamics.

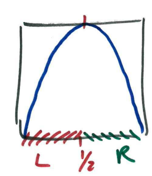
- $g(x) = 4x(1-x)$

→ Let us prove sensit. dep. to ICs.

- Define symbols:

$x_0 \xrightarrow{g} f(x_0) = x_1 \xrightarrow{g} x_2 \xrightarrow{g} x_3$

Symbol L, R



Any orbit : $\{x_0, x_1, x_2, \dots\}$ is associated with the semi-infinite sequence

$S = LRLL\dots = S_1 S_2 S_3 \dots$

where $S_i = \begin{cases} L & \text{if } f^i(x_0) = x_i \in [0, 1/2) \\ R & \text{if } x_i \in [1/2, 1] \end{cases}$

Ex : • $x_0 = 1/3 \longrightarrow \{1/3, 8/9, 32/81, \dots\}$

$\longrightarrow S = LRL\dots$

• $x_0 = 1/4 \longrightarrow \{1/4, 3/4, 3/4, \dots\}$

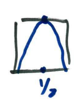
$\longrightarrow S = LRRR\dots = L\overline{R}$

⚠ • $X_0 = 1/2$ is special.

• it can take L or R.

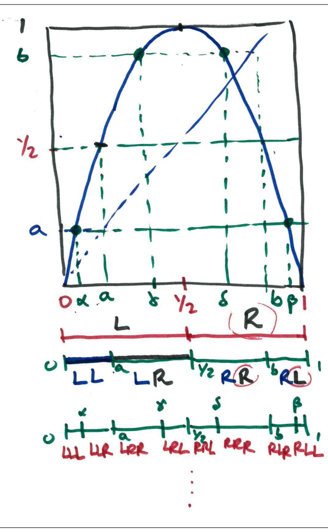
• $x_0 \longrightarrow \{1/2, 1, 0, 0, \dots$

$S = \{{}^{L}_{R}, R, \overline{L}\}$



• Let us avoid 1/2.

Question : what is the symb. dynamics for typical orbits?

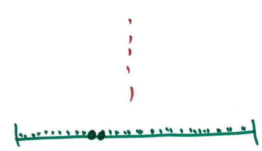


$f(a) = 1/2$

$f(b) = 1/2$

$f(\alpha) = a$, $f(\beta) = a$ } 2 pre-images of a

$f(\gamma) = b$, $f(\delta) = b$ } 2 pre-images of b

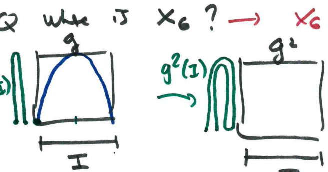


Pattern : • If even R's put •L, •R

• If odd # of R's put •R, •L

Ex : $X_0 \in$ RLLRRRLRLR

Q where is X_6 ? → $X_6 \in [0, 1/2]$

⚠ 

Transition : allowed symbols :

Sensitive dep :

y_0 = LRRLLLLRLRLR...

x_0 = LRRLLLLRLRLL...