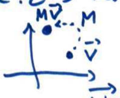
8.1

Sec. 2.3 linear maps



$\mathbb{R}^2 \to \mathbb{R}^2$

$\vec{V}_{n+1} = f(\vec{V}_n) = M \cdot \vec{V}_n$

$\vec{V}_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

<u>Def</u>: A map $A(\vec{v})$ from $\mathbb{R}^m$ to $\mathbb{R}^m$ is linear if $\forall\, \alpha, \beta \in \mathbb{R}$ & $\forall\, \vec{v}, \vec{w} \in \mathbb{R}^m$ we have:

$$A(\alpha\vec{v} + \beta\vec{w}) = \alpha A(\vec{v}) + \beta A(\vec{w})$$

- The origin $\vec{0}$ will ALWAYS be a fixed pt of the map $A(\vec{0}) = \vec{0}$

<u>1D</u>: $f(x) = a \cdot x$, $f'(x) = a$

$\therefore$ Stab. given $f'(0) = a$

$|a| < 1 \Rightarrow S$ & $|a| > 1 \Rightarrow U$

<u>2D</u>: $f(x) = A(x) = A \cdot \vec{x}$

$\longrightarrow$ Stab: check eigenvalues.

8.2

<u>Eigensystem</u>: given matrix A, $\{\lambda, \vec{v}\}$ an eigenpair $\{\lambda$: eigenvalue & $\vec{v}$: eigenvector$\}$ if: $A\vec{v} = \lambda\vec{v}$

$\mathbb{R}$ $\mathbb{R}^2$

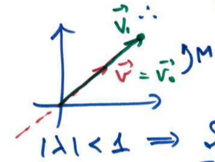
Q: what happens if IC is alligned with an eigenvector?

A: IC: $\vec{V}_0 = \vec{V}$ = eigenvector:

$\underline{n=0}$: $\vec{V}_0 \Rightarrow \underline{n=1}$ $\vec{V}_1 = A\vec{V}_0$

$\vec{V}_1 = A\vec{V}_0 = A\vec{V} = \lambda\vec{v} = \lambda\vec{V}_0$

$\vec{V}_1 = \lambda\vec{V}_0$



$\underline{n=2}$: $\vec{V}_2 = A\vec{V}_1$
$= A(\lambda\vec{V}_0)$
$= \lambda A\vec{V}_0$
$= \lambda\lambda\vec{V}_0$
$= \lambda^2\vec{V}_0$

$\therefore$ $|\lambda| < 1 \Rightarrow S$

$|\lambda| > 1 \Rightarrow U$

8.3

- Find eigenpairs:

$A\vec{V} = \lambda\vec{V}$ find $\begin{pmatrix} x \\ y \end{pmatrix} = \vec{V}$ & λ

$A\vec{V} = \lambda I\vec{V} \Rightarrow [A - \lambda I]\vec{V} = 0$

$\Rightarrow M\vec{V} = 0$

Non-trivial sols $\Rightarrow \det(M) = |M| = 0$

$M = A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix}$

$M = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$

$\det(M) = \underbrace{(a-\lambda)(d-\lambda) - bc}_{\text{quadratic on } \lambda} = 0$ λ_1, λ_2

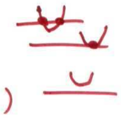
- Evals: Solve $|M| = 0 \Rightarrow \{\lambda_1, \lambda_2\}$
- Evecs: subs. in $A\vec{V}_1 = \lambda_1\vec{V}_1 \Rightarrow$ compute $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

⚠ just need 1 eq. (since both are lin. dep.)

8.4

- From charact. polyn. on λ ($|M| = 0$) we see 3 $\neq$ cases:

1) Distinct REAL evals
2) repeated evals
3) complex evals (comp. conj.)



<u>Ex 2.5</u>: Distinct real evals.

Ex: $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$\det(M) = \begin{vmatrix} a-\lambda & 0 \\ 0 & b-\lambda \end{vmatrix} = (a-\lambda)(b-\lambda) - 0 = 0$

$\lambda_1 = a$, $\lambda_2 = b$

<u>evecs</u>: $\underline{\lambda_1 = a}$: $A\vec{V}_1 = \lambda_1\vec{V}_1$

$\Rightarrow \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = a\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$

$\Rightarrow \begin{cases} a x_1 = a x_1 \Rightarrow x_1 \in \mathbb{R} \\ b y_1 = a y_1 \Rightarrow y_1 = 0 \end{cases}$

$\Rightarrow \vec{V}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\underline{\lambda_2 = b} \Rightarrow \vec{V}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

8.5

Eigenpairs: $\{\lambda_1, V_1\} = \{a, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$

$\{\lambda_2, V_2\} = \{b, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$

<u>IC</u>: alligned with evec.

$\vec{V}_1$: $\vec{V}_1 \xrightarrow{A} \lambda_1\vec{V}_1 \xrightarrow{A} \lambda_1^2\vec{V}_1 \xrightarrow{A} \dots$

$\vec{V}_2$: $\vec{V}_2 \xrightarrow{A} \lambda_2\vec{V}_2 \to \lambda_2^2\vec{V}_1 \to$

<u>In fact</u>:
- $A^n(\vec{V}_1) = \lambda_1^n\vec{V}_1 = a^n\vec{V}_1$
- $A^n(\vec{V}_2) = b^n\vec{V}_2$

<u>Stability</u>:
- $|a| < 1$ & $|b| < 1 \Rightarrow$ contraction $\Rightarrow S$
- any $|a| > 1$ or $|b| > 1 \Rightarrow$ expansion $\Rightarrow U$

Q: what happens if $\vec{V}_0$ is not on $\vec{V}_1$ nor $\vec{V}_2$?

A: Simple: just $\vec{V}_0 = \alpha\vec{V}_1 + \beta\vec{V}_2$

$\Rightarrow A(\vec{V}_0) = A(\alpha\vec{V}_1 + \beta\vec{V}_2) = \alpha A(\vec{V}_1) + \beta A(\vec{V}_2)$

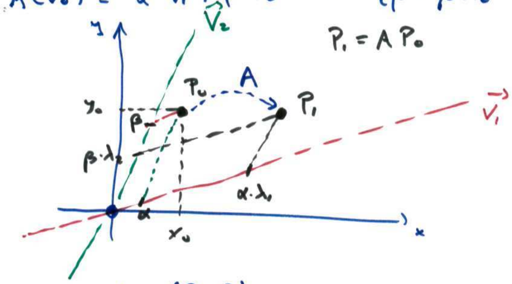
$= \alpha\lambda_1\vec{V}_1 + \beta\lambda_2\vec{V}_2$

8.6

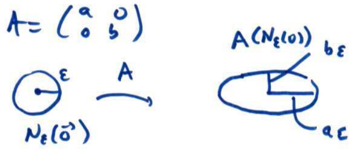
$\therefore$ $\vec{V}_0 = \alpha\vec{V}_1 + \beta\vec{V}_2$

$A(\vec{V}_0) = \alpha'\vec{V}_1 + \beta'\vec{V}_2$ where $\begin{cases} \alpha' = \alpha\lambda_1 \\ \beta' = \beta\lambda_2 \end{cases}$

$P_1 = A P_0$

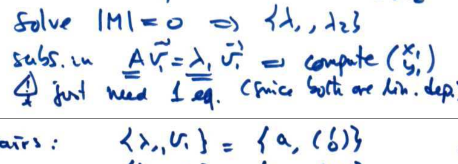


<u>Summary</u>: $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$



8.7

⚠ $|a| > 1$, $|b| < 1 \Rightarrow$ f. pt. is a SADDLE.



<u>Ex: 2.6</u>: repeated evals

$A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \Rightarrow \begin{cases} \lambda_1 = a \\ \lambda_2 = a \end{cases}$

$A\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$: $\begin{pmatrix} x_{n+p} \\ y_{n+p} \end{pmatrix} = A^p \begin{pmatrix} x_n \\ y_n \end{pmatrix}$

So after p-iterates I use $A^p = \underbrace{A \dots A}_{p}$

$\underline{p=1}$: $A^1 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$

8.8

$\underline{p=2}$: $A^2 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^2 & a+a \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} a^2 & 2a \\ 0 & a^2 \end{pmatrix}$

$\underline{p=3}$: $A^3 = A^2 A = \begin{pmatrix} a^2 & 2a \\ 0 & a^2 \end{pmatrix}\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^3 & a^2 + 2a^2 \\ 0 & a^3 \end{pmatrix}$

$= \begin{pmatrix} a^3 & 4a^2 \\ 0 & a^3 \end{pmatrix} = a^2 \begin{pmatrix} a & 3 \\ 0 & a \end{pmatrix}$

$A^2 = a\begin{pmatrix} a & 2 \\ 0 & a \end{pmatrix}$

$A^3 = a^2\begin{pmatrix} a & 3 \\ 0 & a \end{pmatrix} \dots$ $A^p = a^{p-1}\begin{pmatrix} a & n \\ 0 & a \end{pmatrix}$

$A^p = \begin{pmatrix} a^p & p a^{p-1} \\ 0 & a^p \end{pmatrix} = \begin{pmatrix} a^p & 0 \\ 0 & a^p \end{pmatrix} + \begin{pmatrix} 0 & p a^{p-1} \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{A^p} \begin{pmatrix} a^p x + \underline{\underline{p a^{p-1} y}} \\ a^p y \end{pmatrix}$