9.1

Ex 2.7: Complex evals.

let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $\quad a, b \in \mathbb{R},\ b \neq 0$

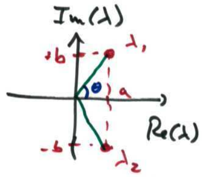
Eval: $\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = 0$

$\Rightarrow (a-\lambda)^2 + b^2 = 0$

$\Rightarrow (a-\lambda)^2 = -b^2$

$\Rightarrow a - \lambda = \pm i b$

$\Rightarrow \boxed{\lambda = a \pm i b}$



⚠ If matrix is real and evals are complex ⟶ they are comp conj.

def: $|\lambda|^2 = a^2 + b^2 = r^2$

$\Rightarrow A = r \begin{pmatrix} a/r & -b/r \\ b/r & a/r \end{pmatrix}$

$\theta = |\arg(\lambda)|$

$\Updownarrow$

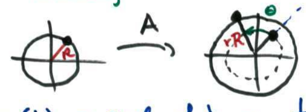
Notice $\left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = 1 \Rightarrow$ def: $\begin{cases} \cos\theta = a/r \\ \sin\theta = b/r \end{cases}$

9.2

$\Rightarrow A = r \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix}$

$= \underbrace{r}_{\text{rescaling}}\ \underbrace{R(\theta)}_{\text{Rotation of ang. } \theta}$

Pic:



$r = \sqrt{a^2+b^2} = |\lambda|$

⚠ $\arg(\lambda)$ = angle of λ = $\tan^{-1}(b/a)$

2.4 Coordinate changes

⟶ ANY matrix in 2D will be equivalent to one of the 3 ex. we saw (2.5, 2.6, 2.7) after a coord. transf.

⟶ Based on "similarity" of matrices ANY matrix (2D) is SIMILAR to one of the 3 examples.

9.3

Def: If S is a non-sing. (i.e. invertible) matrix then $M = S^{-1} A S$ is SIMILAR to the matrix A.

- In fact, on the new coord. frame $S = (\vec{c}_1, \vec{c}_2)$ the matrix $S^{-1}AS$ represents the linear map A in the new basis.

Theo: 2.8: let $A(v)$ be a linear map on $\mathbb{R}^m$ represented by the real, $m \times m$, matrix A (in some coord. syst.). then, for λ_i's = eigenvalues (A):

- the origin is a SINK (attractor) if all λ's: $|\lambda_i| < 1$
- the origin is a SOURCE (repeller) if all λ's: $|\lambda_i| > 1$

9.4

For the proof (2D) we need:

- Similar matrices have SAME evals
- All matrices are similar to one of the following 3 cases:
  1) Ex 2.5: $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ real $\neq$
  2) Ex 2.6: $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ real $=$
  3) Ex 2.7: $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ comp. conj.

$\Longrightarrow M = S^{-1}AS$ (similarity transf. $A \to M$) and A induce SAME dynamical behavior.

⚠ ? What happens when $|\lambda| = 1$

⟶ Stability is inconclusive

⟶ go to higher order when dealing with nonlinear.

Proof: of A & $M = S^{-1}AS$ share eigenvalues:

9.5

$M = S^{-1}AS$, take special case where

$S = [\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n]$ where $\vec{v}_i$ are evecs(A)

- $AS = A[\vec{v}_1 | \dots | \vec{v}_n] = [A\vec{v}_1 | \dots | A\vec{v}_n]$
  $= [\lambda_1 \vec{v}_1 | \dots | \lambda_n \vec{v}_n]$
- $S \cdot \mathrm{diag}(\lambda\text{'s}) = S \cdot \begin{pmatrix} \lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \lambda_n \end{pmatrix}$
  $= [\vec{v}_1 | \dots | \vec{v}_n] \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$
  $= [\lambda_1 \vec{v}_1 | \dots | \lambda_n \vec{v}_n]$

$=$

⚠ Real evals and $\neq$

$AS = S \cdot \mathrm{diag}(\lambda\text{'s})$

$M = S^{-1}AS = \mathrm{diag}(\lambda\text{'s}) = \begin{pmatrix} \lambda_1 & & \emptyset \\ & \ddots & \\ \emptyset & & \lambda_n \end{pmatrix}$

9.6

Def 2.9:

- if $|\lambda_i| \neq 1 \ \forall i$ $\Rightarrow$ A is hyperbolic
- if @ least 1 $|\lambda_i| < 1$ and @ least 1 $|\lambda_j| > 1$ we say the origin is a SADDLE f. pt.

Sec 2.5: Nonlinear Maps & Jacobian

⟶ linear approx tells you the dynamics close to f. pt. $\quad x_n = p + \varepsilon_n$

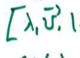
$\boxed{1D}$: $x_{n+1} = f(\underbrace{p + \varepsilon_n}_{x_n}) \approx \underbrace{f(p)}_{p} + \varepsilon_n f'(p)$

$\Rightarrow \underbrace{x_{n+1} - p}_{} \approx \varepsilon_n f'(p)$

$\boxed{\varepsilon_{n+1} \approx f'(p) \cdot \varepsilon_n}$

9.7

$\boxed{2D}$



$\vec{x}_n = \vec{p} + \vec{\varepsilon}_n$

$f \downarrow \ \vec{x}_{n+1} = \vec{p} + \vec{\varepsilon}_{n+1}$

$\vec{\varepsilon}_{n+1} \overset{?}{=} G(\vec{\varepsilon}_n)$

$\vec{x}_{n+1} = f(\vec{x}_n) = f(\vec{p} + \vec{\varepsilon}_n) \quad \Big| \quad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$

$= f(\vec{p}) + (\vec{\varepsilon}_n)_x \cdot \frac{\partial f_1}{\partial x}\Big|_p + (\vec{\varepsilon}_n)_y \cdot \frac{\partial f_1}{\partial y}\Big|_p$
$\quad + (\vec{\varepsilon}_n)_x \cdot \frac{\partial f_2}{\partial x}\Big|_p + (\vec{\varepsilon}_n)_y \cdot \frac{\partial f_2}{\partial y}\Big|_p + O(|\varepsilon|^2)$

$= \underbrace{f(\vec{p})}_{\vec{p}} + Df(\vec{p}) \cdot \vec{\varepsilon} + O(|\varepsilon|^2)$

def: $Df(\vec{p})$ = Jacobian = $\left. \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} \right|_p$

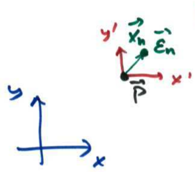
$|\varepsilon| \ll 1 \to |\varepsilon|^2 \approx 0$

$\Rightarrow \vec{x}_{n+1} - \vec{p} = Df(\vec{p}) \cdot \vec{\varepsilon}_n$

9.8

$\boxed{\vec{\varepsilon}_{n+1} = M \cdot \vec{\varepsilon}_n}$, $M = Df(\vec{p})$

↖ linear mapping on the translate ref. frame.



$\begin{cases} \vec{V} = \vec{V}' + \vec{p} \\ \vec{V}' = \vec{V} - \vec{p} \end{cases}$

translation.

Theo 2.11: let f be a map on $\mathbb{R}^m$ and $\vec{p} = f(\vec{p})$ a fixed point, let $\lambda_i = \mathrm{eig}[Df(\vec{p})]$

1- If $|\lambda_i| < 1 \ \forall i \Rightarrow \vec{p}$ is a sink

2- If $|\lambda_i| > 1 \ \forall i \Rightarrow \vec{p}$ is a source

3- @ least $|\lambda_i| > 1$ & @ least $|\lambda_i| < 1$ $\Rightarrow \vec{p}$ is a SADDLE.

4- If a $|\lambda_i| = 1 \Rightarrow$ INCONCLUSIVE.