10.1

Periodic orbits: Stab. in ND is very similar to 1D $\longrightarrow$ multiply Jacobians together.

Theo: Jacobian for period-k orbit: $\{\vec{p}_1, \dots, \vec{p}_k\}$

$$J = Df^k(\vec{p}_1) = Df(p_k) \cdot \dots \cdot Df(p_1)$$

⚠ order is important. In genl. $A \times B \neq B \times A$ for matrizes.

⚠ the product of the Jacobians has evals independent of cyclic permutations of this product!

$\cancel{Df^k(\vec{p}_i) \neq Df^k}$ Evals $(Df^k(\vec{p}_i))$ $=$ Evals $(Df^k(\vec{p}_j))$

⚠ This is NOT true for evectors!!!

10.2

Theo: let f be a map on $\mathbb{R}^m$ and $\{p_1, \dots, p_k\}$ be a period-k orbit

let λ_i = eig $[Df(p_k) \cdot Df(p_{k-1}) \dots Df(p_1)]$

1 - if $|\lambda_i| < 1 \ \forall i \Rightarrow$ period-k SINK

2 - if $|\lambda_i| > 1 \ \forall i \Rightarrow$ period-k SOURCE.

3 - if @ least one $|\lambda_i| > 1$ and $1 |\lambda_j| < 1$ $\Rightarrow$ period-k saddle.

4 - if $|\lambda_i| = 1 \longrightarrow$ linear stab. is inconclusive.

Ex: 2.13: Hénon map $a = 0$, $b = 0.4$
$\longrightarrow$ study f.pts & period-2 & stab.

Hénon: $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - x^2 + by \\ x \end{pmatrix}$

$$\Rightarrow \begin{cases} x_{n+1} = a - x_n^2 + b y_n \\ y_{n+1} = x_n \end{cases}$$

* f.pts $\vec{x} = f(\vec{x}) \Rightarrow \begin{cases} x_{n+1} = x_n \\ y_{n+1} = y_n \end{cases}$

10.3

$$\Rightarrow \begin{cases} a - x^2 + by = x \\ x = y \end{cases} \quad a - x^2 + bx = x$$

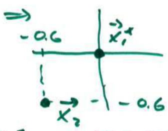
$$\Rightarrow x^2 + (1-b)x - a = 0$$

$$\Rightarrow x_{1/2} = \frac{(b-1) \pm \sqrt{(1-b)^2 + 4a}}{2} = y_{1/2}$$

$a = 0$, $b = 0.4 \longrightarrow \vec{x}_1^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\vec{x}_2^* = \begin{pmatrix} -0.6 \\ -0.6 \end{pmatrix}$

Stab: $Df\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -2x & b \\ 1 & 0 \end{bmatrix}$

$a = 0$, $b = 0.4 \Rightarrow$



$Df\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0.4 \\ 1 & 0 \end{bmatrix}$

$\hookrightarrow \begin{vmatrix} -\lambda & b \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - b = 0$

$$\Rightarrow \boxed{\lambda = \pm\sqrt{b}}$$

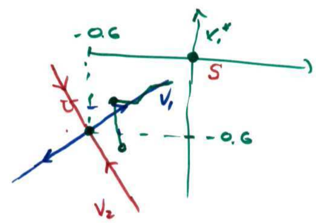
10.4

$b = 0.4 \Rightarrow \lambda_1 = +\sqrt{0.4}$, $\lambda_2 = -\sqrt{0.4} \Rightarrow |\lambda_1, \lambda_2| < 1 \Longrightarrow$ SINK!

$Df\begin{pmatrix} -0.6 \\ -0.6 \end{pmatrix} = \begin{bmatrix} +1.2 & 0.4 \\ 1 & 0 \end{bmatrix}$

$\Longrightarrow \begin{cases} \lambda_1 = 1.472\dots & |\lambda_1| > 1 \\ \lambda_2 = -0.272\dots & |\lambda_2| < 1 \end{cases}$ SADDLE.

Evecs: $V_1 = \begin{pmatrix} 0.8271 \\ 0.5620 \end{pmatrix}$, $V_2 = \begin{pmatrix} -0.2623 \\ 0.9650 \end{pmatrix}$



10.5

Period 2: $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - x^2 + by \\ x \end{pmatrix}$

$$f^2\begin{pmatrix} x \\ y \end{pmatrix} = f\left(f\begin{pmatrix} x \\ y \end{pmatrix}\right) = f\begin{pmatrix} a - x^2 + by \\ x \end{pmatrix}$$

$$= \begin{pmatrix} a - (a - x^2 + by)^2 + b \cdot (x) \\ a - x^2 + by \end{pmatrix}$$

Period-2: $\begin{pmatrix} x \\ y \end{pmatrix} = f^2\begin{pmatrix} x \\ y \end{pmatrix}$

$$\begin{cases} x = a - (a - x^2 + by)^2 + bx \\ y = a - x^2 + by \Rightarrow y = \frac{a - x^2}{1 - b} \end{cases}$$

$$\Rightarrow x = a - \left(a - x^2 + b\,\frac{(a - x^2)}{1 - b}\right)^2 + bx$$

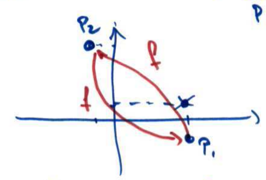
$$\Rightarrow P^4(x) = 0$$

$$\Rightarrow P^2(x)\,(x - x_1^*)(x - x_2^*) = 0$$

$$\Rightarrow \dots \ [\underbrace{x^2 - (1-b)x - a + (1-b)^2}_{P^2(x)}]\,[x - x_1^*]\,[x - x_2^*] = 0$$

10.6

Ex: $a = 0.43$, $b = 0.4$ $\xRightarrow{P^2(x) = 0}$ $\left\{ \underbrace{\begin{pmatrix} 0.7 \\ -0.1 \end{pmatrix}}_{P_1}, \underbrace{\begin{pmatrix} -0.1 \\ 0.7 \end{pmatrix}}_{P_2} \right\}$



$J = \begin{bmatrix} -2x & b \\ 1 & 0 \end{bmatrix}$

Stab: $Df^2(P_1) = Df(P_2) \cdot Df(P_1)$

$$= \begin{bmatrix} -2(-0.1) & 0.4 \\ 1 & 0 \end{bmatrix} *$$

$$= \begin{bmatrix} 0.2 & 0.4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1.4 & 0.4 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

$|$Evals $(Df^2(P_{1/2}))|$ $=$ 0.4, 0.4

$\Rightarrow |\lambda_1, \lambda_2| < 1 \Rightarrow$ SINK.

10.7

Bif: $b = 0.4$ & $a \in [-0.09, \dots, 1.25]$

Ex: T.2.7. (HW)

$$x_{1/2} = \frac{b - 1 \pm \sqrt{(1-b)^2 + 4a}}{2} \longleftarrow \Delta_1$$

$$\{x_{21}, x_{22}\} = x_{2\,1/2} = \frac{1 - b \pm \sqrt{(1-b)^2 - 4(-a + (1-b)^2)}}{2} \quad \Delta_2$$

Existence of f.pts:

Period 1: $\Delta_1 = (1-b)^2 + 4a > 0$

Period 2: $\Delta_2 = \text{- — —} > 0$

Stab: $\longrightarrow$ Jacobians.

$\longrightarrow$ evals $(J) = \{\lambda_1, \lambda_2\}$

S $\iff 0 < |\lambda_1|, |\lambda_2| < 1$

* $\lambda_1 \& \lambda_2 \in \mathbb{R} \Rightarrow \boxed{-1 < \lambda_1, \lambda_2 < 1}$

* $\lambda_1 \& \lambda_2 \in \mathbb{C} \Rightarrow \lambda_1 = \lambda_2^*$, $S \iff |\lambda_1| < 1$