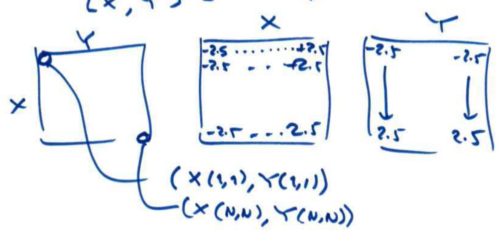
11.1

$|\lambda|^2 = \lambda\lambda^*$ $\qquad |\lambda| = \sqrt{\lambda\lambda^*}$

S $\Rightarrow |\lambda_1|^2 < 1 \Rightarrow \boxed{\lambda_1\lambda_1^* < 1}$

Basic – Matlab – Matrices

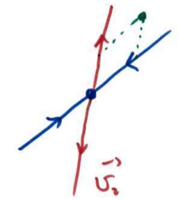
$[X, Y] = \text{meshgrid}(x, y)$



11.2

2.6 Stable & unst. manifolds

First linear: • saddle $\rightarrow$ $\{\lambda_1, \vec{v}_1\}$, $\{\lambda_2, \vec{v}_2\}$

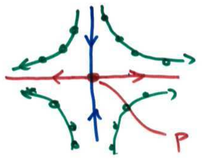


Ex: 2.17: $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2x\\y/2\end{pmatrix}$ two indep. maps.

$x_{n+1} = 2x_n$

$y_{n+1} = y/2$

11.3



$B(p) = \{x = 0\}$

$\leftarrow W^s$: stable manifold

Def: Let be f a map on $\mathbb{R}^2$, let p be a saddle f.pt. (or period saddle).

- The STABLE manifold of p, denoted $W^s(p)$, is the set of pts $\vec{v}$ such that:
$$|f^n(\vec{v}) - f^n(p)| \xrightarrow{n\to\infty} 0$$
- The UNSTABLE manifold, $W^u(p)$,
$$|f^{-n}(\vec{v}) - f^{-n}(p)| \xrightarrow{n\to\infty} 0$$

⚠ $-n \rightarrow$ backwards iterates

- Invert map:

11.4

$$\vec{x}_n \underset{f^{-1}}{\overset{f}{\rightleftarrows}} \vec{x}_{n+1} = f(\vec{x}_n)$$

Algebraically: $\begin{cases} x_{n+1} = f_1(x_n, y_n) \\ y_{n+1} = f_2(x_n, y_n)\end{cases}$

Solve for x_n & y_n:
$$\begin{cases} x_n = g_1(x_{n+1}, y_{n+1}) \\ y_n = g_2(x_{n+1}, y_{n+1})\end{cases}$$

⚠ For linear maps:

$W^s(p)$ = stable eigendirections (line)

$W^u(p)$ = unstable —"— (line)

Q: what about nonlinear maps?

A: locally ($N_\varepsilon(p)$) $\quad W^s \sim$ stab evec, $W^u \sim$ unstab evec.

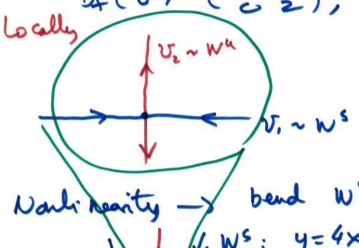
11.5

Ex 2.21: $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x/2\\2y - 7x^2\end{pmatrix}$

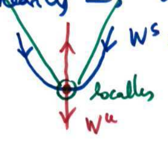
f.pt: $\begin{pmatrix}0\\0\end{pmatrix}$, $\quad Df = \begin{pmatrix}1/2 & 0\\ -14x & 2\end{pmatrix}$

$Df\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1/2 & 0\\0 & 2\end{pmatrix}$, $\quad v_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$\lambda_1 = 1/2$, $\lambda_2 = 2$



- Nonlinearity $\rightarrow$ bend W^u & W^s



11.6

- W^s is an INVARIANT set

INVARIANT: if $\vec{v} \in S \Rightarrow f(\vec{v}) \in S$

To verify W^s: (A) W^s is INVARIANT

(B) $W^s \sim$ stab. evec. locally.

(A) Invariance of the set $y = 4x^2$

- If $V_0 = \begin{pmatrix}x_0\\y_0\end{pmatrix} \in \{y = 4x^2\}$
$\Rightarrow V_1 = f(V_0) \overset{?}{\in} \{y = 4x^2\}$

- $V_0 \in \{y = x^2\} \Rightarrow y_0 = 4x_0^2$

$\Rightarrow V_0 = \begin{pmatrix}x_0\\4x_0^2\end{pmatrix} \xrightarrow{f} V_1 = \begin{pmatrix}x_0/2\\2(4x_0^2) - 7x_0^2\end{pmatrix}$

$\Rightarrow V_1 = \begin{pmatrix}x_0/2\\8x_0^2 - 7x_0^2\end{pmatrix} = \begin{pmatrix}x_0/2\\x_0^2\end{pmatrix} = \begin{pmatrix}x_1\\y_1\end{pmatrix}$

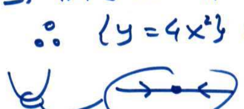
check $y_1 \overset{?}{=} 4x_1^2 \Rightarrow 4x_1^2 = 4\cdot\left(\frac{x_0}{2}\right)^2 = 4\frac{x_0^2}{4}$

11.7

$\Rightarrow 4x^2 = x_0^2 = y_1 \Rightarrow 4x_1^2 = y_1$

$\therefore \{y = 4x^2\}$ is invariant.

(B)



✓

Q: How to find in general W^s

A: Find $\boxed{y = g(x)}$ such that

(A) Invariant

(B) contains f.pt.

(A) Ex: $f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x/2\\2y - 7x^2\end{pmatrix}$

$V_0 = \begin{pmatrix}x_0\\y_0\end{pmatrix}$ ~~$\xrightarrow{f}$ $x_0/2$~~

$= \begin{pmatrix}x_0\\g(x_0)\end{pmatrix} \xrightarrow{f} \begin{pmatrix}x_0/2\\2g(x_0) - 7x_0^2\end{pmatrix} = \begin{pmatrix}x_1\\y_1\end{pmatrix}$

11.8

$y_1 = g(x_1) \Rightarrow \boxed{2g(x_0) - 7x_0^2 = g(x_0/2)}$ $\forall x_0$

$$2g(x) - g\left(\tfrac{x}{2}\right) - 7x^2 = 0 \quad (*)$$

$\leftarrow$ functional eq.

!! $\rightarrow$ funct. eqns. cannot be solved in general.

$\rightarrow$ Numerics.

check: $y = 4x^2$ should satisfy (*)

$$2\cdot 4x^2 - 4\left(\tfrac{x}{2}\right)^2 - 7x^2$$
$$= 8x^2 - x^2 - 7x^2 = (8-8)x^2 = 0 \checkmark$$