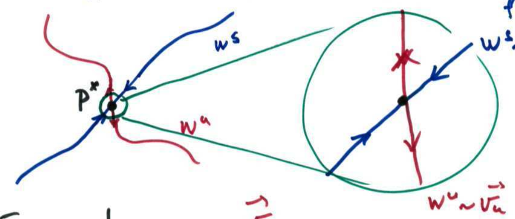
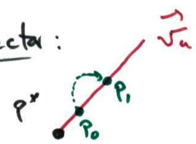
Numerical method to obtain W^u / W^s

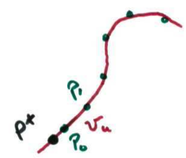
→ Exploit the fact that $W^u \sim \vec{v}_u$ & $W^s \sim \vec{v}_s$ locally (i.e. close to f. pt.)



Eigenvector:



If $P_0 \in \vec{v}_u \Rightarrow P_1 = f(P_0) \in \vec{v}_u$

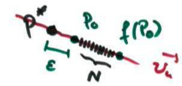


$\therefore f^n(P_0) \in W^u$ If $P_0 \in \vec{v}_u$ and $|P_0 - P^*| \ll 1$

→ Take N (large $\approx$ 1000's) pts between P_0 & P_1 → N orbits that will span W^u.

→ Set of ICs: $I = [\underbrace{P^* + \varepsilon \vec{v}_u}_{P_0}, f(P_0)]$, $|\varepsilon| \ll 1$ (10^{-6})

take $N_\#$ pts on I & iterate 30 times.



- Need to this 4 times:

(1) $P_0 = P^* + \varepsilon \vec{v}_u$ (2) $P_0 = P^* - \varepsilon \vec{v}_u$

(1) + (2) ⟹ W^u

(3) INV. of Map: $W^u \longleftrightarrow W^s$

Same as (1) & (2) for f^{-1} ⟹ W^s of f

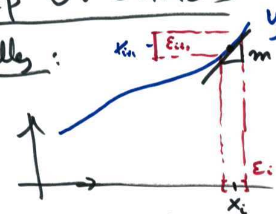
Find inverse:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = F\begin{pmatrix} x_n \\ y_n \end{pmatrix} \Rightarrow \begin{cases} x_{n+1} = f(x_n, y_n) \\ y_{n+1} = g(x_n, y_n) \end{cases}$$

$$\text{Inv}: \begin{pmatrix} x_n \\ y_n \end{pmatrix} = F^{-1}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} \Rightarrow \begin{cases} x_n = t(x_{n+1}, y_{n+1}) \\ y_n = w(x_{n+1}, y_{n+1}) \end{cases}$$

Chap 3: CHAOS

Locally:



$m = f'(x_i)$ → gives expansion (or contraction) of orbits close to x_i.

Chaos "⟷" expansion → Lyapunov Exponents.

3.1: Lyap. exp.

orbit: $\{x_1, x_2, \dots\}$ @ each pt. we have an expansion $= |f'(x_i)|$

After k iterates expansion rate: $|f'(x_1)| \cdot |f'(x_2)| \dots |f'(x_k)|$

Def. 3.1: Let f be a smooth map on $\mathbb{R}$. The LYAPUNOV NUMBER $L(x_1)$ of the orbit $\{x_1, x_2, \dots\}$ is:

$$L(x_1) \equiv \lim_{n\to\infty} \left[|f'(x_1)| \dots |f'(x_n)| \right]^{1/n}$$

$L(x_1)$: geometric averg. of expansion rates.

If the limit exists. And we define the LYAPUNOV EXPONENT:

$$\lambda(x_1) = h(x_1) = \ln(L(x_1))$$

$$\Rightarrow \boxed{\lambda(x_1) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln|f'(x_i)|}$$

- $L > 1$ ⟹ $\lambda > 0$ ⟹ EXPANSION!
- $0 \leq L < 1$ ⟹ $\lambda < 0$ ⟹ CONTRACTION!

⚠ If $f'(x_j) = 0$ for some j ⟹ Lyap. exp. NOT defined.

Ex: T3.1: for f: $L(x_1) = e$ ⟹ for f^k: $L(x_1) = e^k$

Periodic orbit, of period k:

$$\lambda(x_1) = \frac{\ln|f'(x_1)| + \dots + \ln|f'(x_k)|}{k}.$$

Def: 3.3: Let f be a smooth map. An orbit $\{x_1, x_2, \dots\}$ is called ASSYMPTOTICALLY PERIODIC if it converges to a periodic orbit as $n \to \infty$

I.e. there exist a periodic orbit $\{y_1, \dots, y_k\}$ such that $\lim_{n\to\infty} |x_n - y_n| = 0$

⚠ an eventually periodic orbit is also an assymptotically periodic orbit.

Thm: 3.4 If $\{x_1, \dots\}$ satisfying $f'(x_i) \neq 0$ $\forall i$ and is assymp. periodic to $\{y_1, \dots\}$ then the two orbits share Lyap. exp. & Lyap. #.

3.2 Chaotic orbits

Def 3.5: Let f be a map on $\mathbb{R}$ and let $\{x_1, \dots\}$ be a bounded orbit. The orbit is called CHAOTIC if:

1.- $\{x_1, \dots\}$ is NOT assym. periodic (not event. periodic or periodic)

2.- The Lyap Exp. $\lambda(x_1) > 0$

Problem: remove x's that touch 1/2 or are assymp. periodic.

Because of expansion: ALL assymp. per. orbits need to be eventually periodic.

Follow all periodic orbits

A → Write orbit in binary:

$$x = \{.b_1 b_2 b_3 \dots\}$$

$$x = \sum_{i=1}^{\infty} b_i 2^{-i} = b_1 \tfrac{1}{2} + b_2 \tfrac{1}{4} + b_3 \tfrac{1}{8} + \dots$$

Ex: $\frac{1}{2} = 1 \cdot \frac{1}{2} + 0 \frac{1}{4} + 0 \frac{1}{8} + \dots$

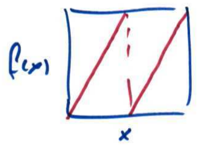
$= \{.1000\dots\} = \{.1\bar{0}\}$

$\frac{1}{4} = \{.01\bar{0}\}$

$1/5 = 1/8 + 1/16 + 1/32 + \dots = \{.\overline{0011}\}$

Ex 3.6: Compute Lyap. exp. for the map: 12.8

$$f(x) = 2x \pmod 1$$



- If x_0 ~~does~~ never touches $1/2$ then
  $$\lambda(x_0) = \ln 2$$
  [derivative is always $= 2$]
- If $\{x_0, \ldots\}$ is NOT ~~eventually~~ asymp. periodic

  $\Longrightarrow$ CHAOS.