13.1

What does f do to binary represent?

$$x = \sum_{i=1}^{\infty} b_i 2^{-i}$$

$\boxed{f}$: $f(x) = 2\left(\sum_{i=1}^{\infty} b_i 2^{-i}\right) \pmod 1$

(a) If $0 \le x \le 1/2$ $(b_1 = 0)$

$$\Rightarrow f(x) = 2\sum_{i=2}^{\infty} b_i 2^{-i} = \sum_{i=2}^{\infty} b_i 2^{-i+1}$$

$$= \sum_{i=1}^{\infty} b_{i+1} 2^{-i} = .b_2 b_3 b_4 \ldots$$

(b) If $1/2 \le x < 1$ $(b_1 = 1)$

$$\Rightarrow f(x) = 2\left(\sum_{i=1}^{\infty} b_i 2^{-i}\right) \pmod 1$$

$$= 2\left(b_1 \frac{1}{2} + \sum_{i=2}^{\infty} b_i 2^{-i}\right) - 1$$

$$= \cancel{1} + \sum_{i=2}^{\infty} b_i 2^{-i} \cancel{-1}$$

$$= \sum_{i=1}^{\infty} b_{i+1} 2^{-i} = .b_2 b_3 b_4 \ldots$$

13.2

∴ f just shifts the seq. to left by one.

$$f(.b_1 b_2 \ldots) = .b_2 b_3 \ldots$$

∴ Any eventually periodic orbit of period k will have a seq:

$$.b_1 b_2 \ldots b_n \overline{(b_{n+1} b_{n+2} \ldots b_{n+k})}$$

n & k need < ∞

∴ a) If x is RATIONAL ⇒ orbit is eventually periodic

b) If x is IRRATIONAL ⇒ orbit is not eventually periodic.

13.3

Q: How do we know that x irrational does not give an orbit that is asymptotically periodic.

A: ANY periodic orbit will ~~have~~ be UNSTABLE!!!

$$\Rightarrow f^{k\prime}(x) = 2^k > 1$$

⇒ Nothing can asymptote to them.

∴ 1) $\lambda = \ln(2) > 0$

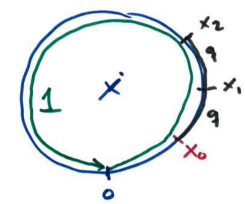
2) No asympt. periodic

⇒ CHAOS for x in IRRATIONALS

13.4

Ex 3.7: $f(x) = (x + q) \pmod 1$

q is IRRATIONAL



⇒ No orbit is periodic.

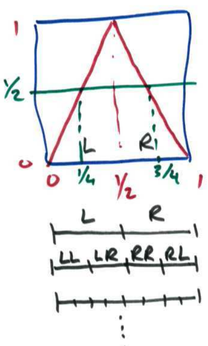
$f'(x) = 1 \Rightarrow \lambda = \ln(1) = 0$

⇒ No chaos

Def: A bounded orbit that is not asymp. periodic that does not display sensitive dependence to ICs $(\lambda \neq 0)$ ⇒ QUASIPERIODIC ORBIT

13.5

Tent map: $T(x) = \begin{cases} 2x & x \le 1/2 \\ 2(1-x) & 1/2 < x \end{cases}$



Symbolic dynamics {L, R}

⇒ Same symb. as logistic map but intervals are of length 2^{-k}

• $|f'(x)| = 2$

$\Rightarrow \lambda = \ln 2$

13.6

Theo 3.9: The tent map has ∞-tely many chaotic orbits.

proof:
- $|f'| = 2$ except $x = 1/2$
- $x = 1/2$ → rationals, No need of them
- every periodic orbit → UNSTABLE ⇒ no asymp. periodic orbits

⇒ CHAOS.

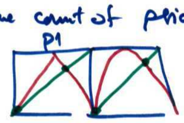
3.3 Conjugacy $\overbrace{\text{tent map}}^{T} \longleftrightarrow \overbrace{\text{logistic map}}^{G}$

Observations:
- T & G ∞-tely many chaotic orbits
- T & G same sym. dyn.
- T & G ⇒ $\lambda = \ln 2$

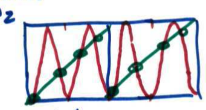
13.7

• Same count of periodic orbits

P1



P2



... ⇒ Same for ∀ P.

• Stability.

	G	T
* fp (P1)	$G' = \pm 2$	$T' = \pm 2$
* P2	$G'(x_1)G'(x_2) = -4$	$T'(x_1)T'(x_2) = -4$
	⋮	⋮

∴ these two maps T & G seem to share ALL dynamical properties!!!

13.8

Def: 3.10: The maps f & g are CONJUGATE if there exist a continuous, 1-to-1, map C(x) such that

$$C \circ f = g \circ C$$

$$C(f(x)) = g(C(x))$$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "C"'] & f(x) \arrow[d, "C"] \\ C(x) \arrow[r, "g"] & C(f(x)) = g(C(x)) \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "C"'] & f(x) \\ {} \arrow[r, "g"] & {} \arrow[u, "C^{-1}"'] \end{tikzcd}$$

$C \circ f = g \circ C$

$\Rightarrow C \circ f \circ C^{-1} = g \circ \underline{C \circ C^{-1}}$

$\Rightarrow g = C \circ f \circ C^{-1}$

• $C(x) = \dfrac{(1-\cos\pi x)}{2}$ is a conjugacy between T & G 13.9

check : $C\circ T = G\circ C$

* $G(C(x)) = 4\,C(x)\,(1-C(x))$

$= 4\,\dfrac{1-\cos\pi x}{2}\left(\dfrac{2}{2} - \dfrac{1-\cos\pi x}{2}\right)$

$= 4\cdot\dfrac{1}{2}\dfrac{1}{2}(1-\cos\pi x)(1+\cos\pi x)$

$= 1-c^2 = \sin^2\pi x$

* $C(T(x)) = \dfrac{1-\cos\pi(T(x))}{2}$

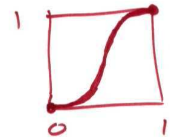
$x<1/2$: $= \dfrac{1-\cos\pi(2x)}{2} = 1-\cos$

$x>1/2$: $= \dfrac{1-\cos\pi(2(1-x))}{2} = \dfrac{1-\cos(\cancel{2\pi}+2\pi x)}{2}$

$\cos 2a = 1-2\sin^2 a$

$\Rightarrow C(T(x)) = \dfrac{1-\cos(2\pi x)}{2} = \dfrac{1-[1-2\sin^2(\pi x)]}{2}$

$\therefore\ G\circ C = C\circ T$ for $C(x) = \dfrac{1-\cos\pi x}{2}$ 13.10



$C(x)$.

$$\begin{array}{ccc} x & \xrightarrow{T} & T(x) \\ {\scriptstyle C}\downarrow & & \downarrow{\scriptstyle C} \\ y & \xrightarrow{G} & C(T(x)) = G(C(x)) \\ & & = G(y) \end{array}$$

$\boxed{y = C(x)}$

$\Rightarrow \boxed{T(x) = C^{-1}(G(y))}$