Periodic pts: $G^n(x) = \underbrace{CTC^{-1}\, CTC^{-1}}\cdots CTC^{-1}$

$$\boxed{G^n = C\,T^n\,C^{-1}}$$

$$\Rightarrow \boxed{T^n = C^{-1} G^n C}$$

$\therefore$ G & T share all periodic orbits.

Stability: $G(C(x)) = C(T(x))$

$\overset{C'}{\Longrightarrow}$ $C' \cdot G'(C(x)) = T'(x)\, C'(T(x))$

Suppose x is f.pt. of T: $T(x) = x$

$\Rightarrow$ $C' \cdot G'(C(x)) = T'(x)\, C'(x)$

$\therefore x \neq 0, 1 \Rightarrow C' \neq 0 \Rightarrow G'(C(x)) = T'(x)$

$\Rightarrow \boxed{G'(y) = T'(x)}$

$\therefore$ stab. for f. pts in G & T is SAME

Period n orbit:

n=2 : • $T^2(x) = x$

• $G^2 = C\,T^2\,C^{-1}$ #1 [illegible]

$\overset{C'}{\Longrightarrow}$ $G^2(C(x)) = C(T^2(x))$

$\Longrightarrow$ $C'(x) \cdot [G^2]'(C(x)) = [T^2]' \cdot C'(\underset{x}{T^2(x)})$

If $x \neq 0,1 \Rightarrow C' \neq 0 \Rightarrow$

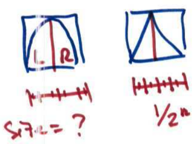
$[G^2]'(C(x)) = [T^2]'(x)$

$$\boxed{[G^2]'(y) = [T^2]'(x)}$$

Same can be done for any n:

$$\boxed{[G^n]'(y) = [T^n]'(x) \quad \text{if } T^n(x) = x,\ y = C(x)}$$

$\therefore$ Periodic orbits of G (log. map) are ALL unstable [exp rate 2^k]



Size of k-intervals for G

T: $s_k = x_2 - x_1 = \frac{1}{2^k}$

G:

$|\underbrace{C(x_2)}- C(x_1)| = \left|\int_{x_1}^{x_2} C'(x)\,dx\right| = \left|\int_{x_1}^{x_2} \frac{\pi}{2}\sin \pi x\, dx\right|$

$s_k^{(G)} = |y_2 - y_1| \leq \frac{\pi}{2}\int_{x_1}^{x_2} 1 = \frac{\pi}{2}(x_2 - x_1) = \frac{\pi}{2} s_k$

$\therefore |y_2 - y_1| = \frac{\pi}{2}\frac{1}{2^k} = \boxed{\frac{\pi}{2^{k+1}}}$

Lyap. Exp. for G: Lyap exp. $T = \lambda = \ln 2$

Consider $\{x_1, x_2, \ldots\}$ an orbit for T:

$[T^k(x_1)]' = T'(x_k)\,T'(x_{k-1}) \cdots T'(x_1)$ ①

but: $GC = CT \Rightarrow C'G'(C) = T'C'(T)$

$\Rightarrow T'(x_k) = \frac{C'(x_k) \cdot G'(C(x_k))}{C'(\underbrace{T(x_k)}_{x_{k+1}})}$ ②

② in ① $\Rightarrow [T^k(x_1)]' = \frac{\cancel{C'(x_k)} \cdot G'(C(x_k))}{C'(x_{k+1})} \times$

$\times \frac{\cancel{C'(x_{k-1})} \cdot G'(C(x_{k-1}))}{C'(\cancel{x_k})} \times \cdots \times \frac{C'(x_1) \cdot G'(C(x_1))}{\cancel{C'(x_2)}}$

$\Rightarrow [T^k(x_1)]' = \frac{C'(x_1)}{C'(x_{k+1})}\, G'(C(x_k))\, G'(C(x_{k-1})) \cdots G'(C(x_1))$

$= \quad \text{''} \quad G'(y_k)\, G'(y_{k-1}) \cdots G'(y_1)$

$= \frac{C'(x_1)}{C'(x_{k+1})}\,[G^k(y_1)]'$

Lyap. Exp.: $\Sigma \ln|T'(x_k)| = \ln \Pi(T'(x_k))$

$= \ln\left[\frac{C'(x_1)}{C'(x_{k+1})}\, \Pi\, G'(y_k)\right] = \ln\left(\frac{C}{C}\right) + \ln \Pi G'$

$\lambda_{\text{for } T} = \lim_{k\to\infty} \frac{1}{k}\sum_{i=1}^{k} \ln|T'(x_i)|$

$= \lim_{k\to\infty} \frac{1}{k}\Big[\underbrace{\ln \cancel{C'(x_1)}^{\,0} - \ln \cancel{C'(x_{k+1})}^{\,0}}_{x_i \neq 0,1\ \forall i} + \sum_{i=1}^{k} \ln|G'(y_i)|\Big]$

$= \lim \frac{1}{k}\Sigma \ln|G'(y_i)| = \lambda$ for G.

$\therefore$ Lyap. Exp. of G is $\ln 2$ if $x_i \neq 0,1$

$\forall y$ such $y = C(x)$ & x is an irrational between (0,1) gives rise to a non-periodic orbit in G with $\lambda = \ln 2$ $\therefore$ CHAOTIC

Dense orbits:

Def: 3.14: Let A be a subset of B. The set A is said to be dense in B if arbitrarily close to each pt in B there is a pt. in A.

i.e. $\forall x \in B \Rightarrow \exists y \in N_\varepsilon(x)$ for $\forall \varepsilon > 0 \mid y \in A$

Ex: • Rationals are dense on [0,1]

• Irrationals are dense on [0,1]

Thm: Chaotic orbits of G are dense on [0,1]

Proof: Just construct the right orbit.

- Take symb. dyn. description {L,R}
- Transition graph is complete

- construct seq. that has ALL possible seq.

$S = \{L \mid R \mid LL, LR, RL, RR \mid LLL, LLR, LRL, \ldots \mid$

$\ldots \ldots \mid$

bla .. bla ∎

3.4 Transition graphs

→ Not covering

→ Most important result:

"Period 3 implies chaos"

- details: Challenge #1 p32
- Also Sharkovskii's theo challenge #3 p135.