3.5 Basins of attraction

15.1

Def 3.21 Let f be a map on $\mathbb{R}^n$ and let p be an attracting f. pt. (or periodic orbit). The BASIN OF ATTRACTION is the set of ALL pts such that $|f^k(x) - f^k(p)| \xrightarrow[k\to\infty]{} 0$

Ex: In $\mathbb{R}^m$ a linear map with $|\text{evals}| < 1 \implies B(\vec{0}) = \mathbb{R}^m$

Theo 3.23: Let f be a cont. map on $\mathbb{R}^1$

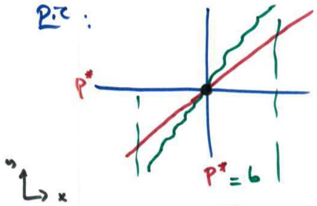
(1) If $f(b)=b$ & $x < f(x) < b$

(2) If $f(b)=b$ & $b < f(x) < x$

$\implies f^k(x) \xrightarrow{k\to\infty} b$

15.2

Pic:



Proof:

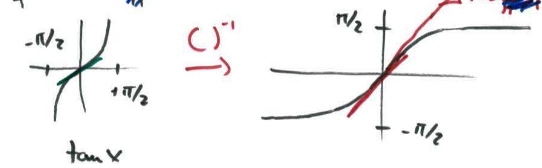
(1) If $x < b \implies f \nearrow$

$\{x_1, x_2, \dots\}$ is an $\nearrow$ seq. & bounded by $x=b$

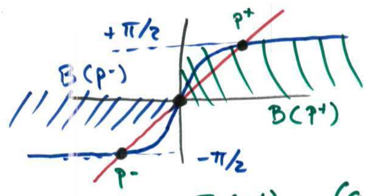
$\implies$ seq. must conv.

no other f. pt. $\implies$ seq. conv. to $x=b$

Ex: $f(x) = 2 \tan^{-1} x$



15.3



$B(p^+) = (0, +\infty)$

$B(p^-) = (-\infty, 0)$

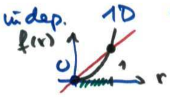
"$B(0) = 0$"

Ex: 2D, $f(r,\theta) = (r^2, \theta - \sin\theta)$ $= (x^2+y^2, \tan^{-1}\frac{y}{x} - y)$

$\begin{cases} r_{n+1} = r_n^2 \\ \theta_{n+1} = \theta_n - \sin\theta_n \end{cases}$

2 indep. 1D maps.

$\boxed{r}$:

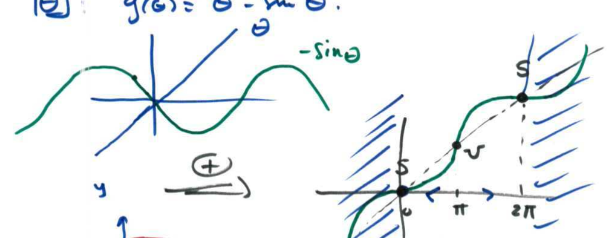


- $r^*=1$ is f. pt
- $r^* = 1$ is U.

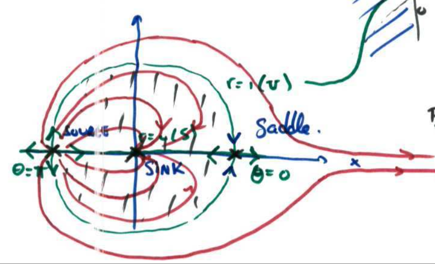
15.4

$r^* = 0$ is a S f. pt. & $B(0) = [0,1)$

$\boxed{\theta}$: $g(\theta) = \theta - \sin\theta$.



$B(0) = [0, 2\pi] \setminus \{\pi\}$



$B(0)$ = unit disk without its circumference.

15.5

Def 3.27: Schwarzian

$$S[f(x)] = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

Theo: 3.29 (1D)

If f has a negative schwarzian and p is a f. pt. or periodic orbit then either:

1- p has an infinite basin.

or 2- there is a crit. pt. ($f'(x)=0$) of f in the basin of p.

or 3- p is a source.

Ex: prove that all periodic orbits of log. map are unstable.

$S[f(x)] = \dots = -\frac{3}{2}\left(\frac{-2a}{a-2ax}\right)^2 < 0$

15.6

- orbits outside $[0,1] \to -\infty$

  $\therefore$ period orbits in $[0,1]$ do NOT have an ∞ basin. $\to$ [NOT 1-]

- Crit. pts: $x_c = 1/2$

  $f(x_c) = f(1/2) = 1 \xrightarrow{f} 0$

  $x_c \in B(0)$

- any other f. pt. or periodic orbit ($\neq 0$) $\implies$ SOURCE by [3-]

15.7

CHAP 4 FRACTALS

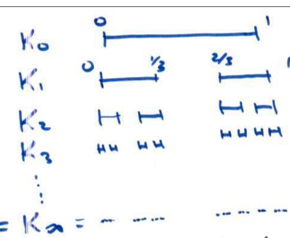
- Conventional (i.e. non fractal) object becomes "boring" after magnification $\hookrightarrow$ •, /, ▱, ⬡
- Fractals have prop. of keeping complexity after magnification
- No univ. def. ☹
- Fractal has a dimension that is fractional (i.e. not integer)

4.1 Cantor set $\longrightarrow$ easiest of all fractals

Take $[0,1]$ remove middle 1/3

reapply to each piece.

15.8



Middle 1/3 Cantor set

Length:

$L(K_0) = 1$

$L(K_1) = 2 \cdot 1/3$

$L(K_2) = 2^2 \cdot \frac{1}{3^2} = (2/3)^2$

$L(K_3) = 2^3 \cdot 1/3^3 = (2/3)^3$

$\vdots$

$L(K) = \lim_{n\to\infty} (2/3)^n = 0$

Measure: A set S is said to have measure 0 if it can be covered with intervals whose total length is arbitrarily small.

Ex: • a point : ⊙

- Any FINITE collection of point has measure 0
- set of rationals has measure 0
- set of irrationals has measure 1.

Back to K : → base-3 representation

Take $0 \le r \le 1 \Rightarrow r = a_1 \cdot \frac{1}{3} + a_2 \cdot \frac{1}{9} + a_3 \cdot \frac{1}{3^3}$

$$r = \sum_{k=1}^{\infty} a_k 3^{-k} = .a_1 a_2 a_3 \ldots$$

where $a_i = \{0, 1, 2\}$

⚠ description is NOT unique:

Ex: $\frac{1}{3} = \{0.10000\} = \{.0\bar{2}\}$

× ✓

K_1: 0 —— 1/3   2/3 —— 1

If $r \in K_1$ → $a_1 = 0$, $a_1 = 2$

$\frac{1}{3} = \{.\cancel{10}\} = \{.0\bar{2}\}$

K_n: n-th symbol needs to be $\{0,2\}$

∴ K is set of all $.a_1 a_2 \ldots$ such that $a_i \in \{0,2\}$

4.2 Probabilistic construction of Fractals

Iterated function system (IFS)

"def. a map with ≠ maps and choose between them randomly"