

Ex: Middle-Third Cantor set:

- start with $x_0 \in [0, 1]$
- flip coin
 - * heads: move $\frac{1}{3}$ of the way towards $\frac{1}{2}$
 - * tails: $\dots - \frac{1}{3} - \frac{2}{3} - \frac{1}{3} - \dots$
- = repeat

Q: where does x_0 go to?

$$A: x_0 \rightarrow K$$

Iterated Function Systems (IFS)

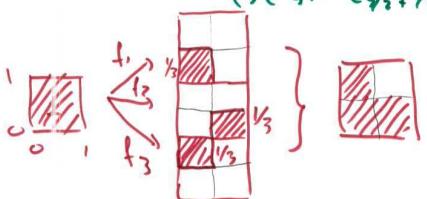
$$\text{IC: } x_0 \in [0, 1] : \begin{cases} f_1(x) = \frac{2}{3}x + \frac{x}{3} \\ f_2(x) = \frac{x}{3} \end{cases}$$

Q: Where is x_0 mapped?

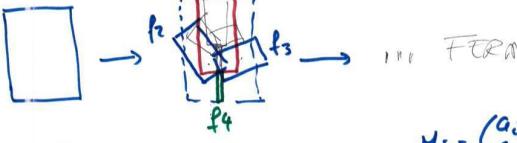
Def 4.4: An IFS on \mathbb{R}^m is a collection of r maps: $\{f_1, \dots, f_r\}$ of \mathbb{R}^m together with their respective probabilities $\{p_1, \dots, p_r\}$. $[\leq p_i = 1]$

Ex: Sierpinski triangle (2D)

$$\text{IFS: } \begin{aligned} f_1(S) &= \left(\frac{x}{2}, \frac{y}{2} \right), P_1 = \frac{1}{3} \\ f_2(S) &= \left(\frac{x}{2}, \frac{y}{2} + \frac{1}{2} \right), P_2 = \frac{1}{3} \\ f_3(S) &= \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2} \right), P_3 = \frac{1}{3} \end{aligned}$$



Ex: Fern: 2D + 4 f's



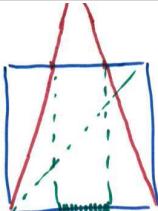
$$f_i(y) = M_i(x) + V_i \quad M_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad V_i = \begin{pmatrix} e_i \\ f_i \end{pmatrix}$$

4.3 Fractals from deterministic systems

$$\text{ Tent map: } T_a(x) = \begin{cases} ax & x \leq \frac{1}{2} \\ a(1-x) & x > \frac{1}{2} \end{cases}$$

$$a=2: \quad \begin{array}{c} \square \\ \rightarrow \end{array} \quad \begin{array}{c} \triangle \\ \square \end{array} \quad 1 < a < 2 \quad \begin{array}{c} \triangle \\ \square \end{array}$$

$f_3(x)$:

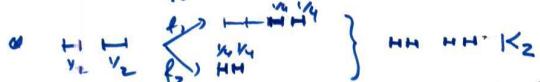
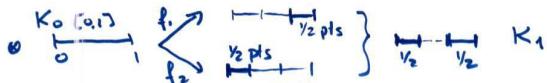


\therefore * All ICs that $\in K$ will never leave $[0, 1]$

* All ICs $x_0 \in [0, 1] \setminus K$ will go to Hell!
Complement of K

If $x_0 \in K \Rightarrow \{x_0, x_1, \dots\}$ is a chaotic orbit with $\lambda = \ln 3$!!!

16.1



$\dots \xrightarrow{\infty} K : \text{Cantor set.}$

+ Any IC $\rightarrow K$

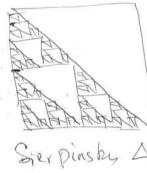
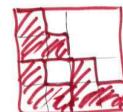
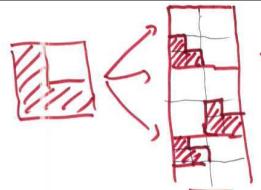
+ If $x_0 \in K \rightarrow x_i \in K \forall i$

+ If $x_0 \notin K \rightarrow \{x_i\}_{i=1}^{\infty}$ tends to K asymptotically.

• At stage K_k : you'll be @ most $\frac{1}{6} \cdot \frac{1}{3^k}$ apart from a pt. in K :

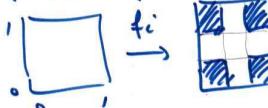
$$\begin{aligned} &\frac{1}{3^k} \quad \frac{1}{3^k} \quad \dots \quad \frac{1}{3^k} \\ &V_k = \frac{1}{3} \cdot \frac{1}{3^{k-1}} \quad \frac{1}{3} \cdot \frac{1}{3^{k-1}} = \frac{1}{3} \cdot \frac{1}{3^k} \quad \dots \quad \frac{1}{3} \cdot \frac{1}{3^k} \text{ from } K_k \\ &k=1, \frac{1}{6} \text{ from } K_1 \quad k=2, \frac{1}{18} \text{ from } K_2 \end{aligned}$$

16.2



Sierpinski Δ

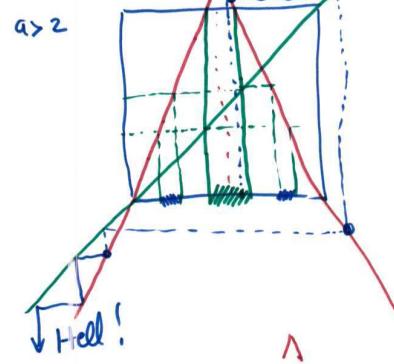
Ex: Sierpinski carpet.



16.4



$a=1$ $0 < a < 1$

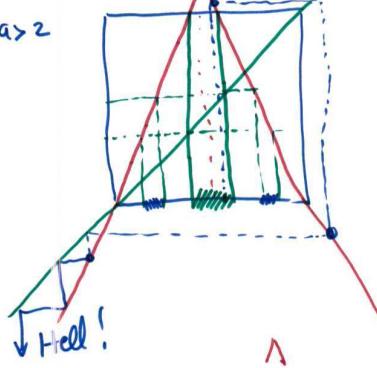


16.5

16.5



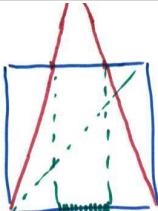
$a > 2$



16.6

16.7

$f_3(x)$:



\therefore * All ICs that $\in K$ will never leave $[0, 1]$

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Complement of K

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