4.4 Fractal Basin boundaries

+ pic p 165 of book
  → Basins of attractor can have a fractal structure.

Julia sets & Mandelbrot sets

Consider the following 2D map:

$$P_c(z) = z^2 + c \qquad z, c \in \mathbb{C}$$
$$c = a + ib$$

$$z_0 = x_0 + i y_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$
$$z_1 = P(z_0) = z_0^2 + c = (x_0 + i y_0)^2 + a + ib$$
$$= x_0^2 - y_0^2 + 2i x_0 y_0 + a + ib$$
$$= [x_0^2 - y_0^2 + a] + [2x_0 y_0 + b]\, i = \begin{pmatrix} x_0^2 - y_0^2 + a \\ 2x_0 y_0 + b \end{pmatrix}$$

2D: $\begin{cases} x_{n+1} = x_n^2 - y_n^2 + a \\ y_{n+1} = 2x_n y_n + b \end{cases}$

c=0 : $a = 0 = b$

$$P_0(z) = \boxed{z^2}$$

f.pt : $\underbrace{z^2 = z}$ $\Rightarrow$ $z_1^* = 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $z_2^* = 1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

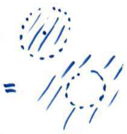
$$J = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

stab : $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $J\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = 0$ superstable sink.

stab : $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $J\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1 = \lambda_2 = 2$ SOURCE.

Map : $z = re^{i\theta} \xrightarrow{P_0} z^2 = \underline{r^2} e^{\underline{i2\theta}}$

P_0^n : ⊘ → •



$B(0) =$, $B(\infty) =$

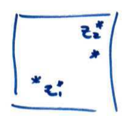
$B(O) = O$ ← jump chaotically on the unit circle.

Q: What is the shape of Basins of attraction for ≠ c's?

$c \neq 0$:

f.pt : $P_c(z) = z \Rightarrow z^2 + c = z$

$$\Rightarrow z^2 - z + c = 0$$
$$z^*_{1/2} = \frac{1 \pm \sqrt{1-4c}}{2}$$



Thm (Fatou): Every attracting cycle (periodic or f.pt) for a rational map (quotient of polynomials) attracts @ least one critical pt of P ($P' = 0$).

⚠ This is a generalization of thm 3.29 using Schwarzian.

⚠ for $P_c = z^2 + c \longrightarrow 1$ crit. pt.
→ at MOST 1 stable cycle.

⚠ Sometimes $P_c(z)$ does not have attracting cycles!

ex: $c = -i$, crit. pt $= 0$

$0 = P_i(0) = 0^2 - i = \underline{-i} \xrightarrow{P} (-i)^2 - i = \underline{-1-i}$
$\xrightarrow{P} (-1-i)^2 - i = \cancel{1} - \cancel{1} + 2i - i = \underline{i}$
$\xrightarrow{P} (i^2) - i = \underline{-1-i}$

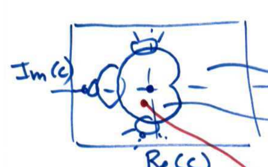
∴ Eventually p-2 orbit $\{i, -1-i\}$

→ Stab .... UNSTABLE.

∴ crit. pt. "absorbed" by U ⇒ No stab. cycles!

Mandelbrot set: Q: Where does $z_0 = 0$ go under P_c?

↳ $M = \{ c \mid z_0 = 0$ is not in $B(\infty)$ for $P_c(z) = z^2 + c \}$



$0 \in M$

c such that $P^{\infty}(z_0) = \infty$

c such that $z_0 \not\to \infty$

Julia Set: Take a $c \in M$ and draw pts that do NOT go to ∞ ≡ Julia set is the BOUNDARY of $B(\infty)$ and the rest.

Ex: $c = -0.17 + 0.78i \in M$

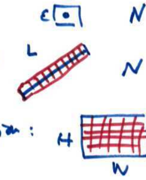
4.5 Fractal dimension

Measure how many ε-balls we need to cover an object. Use ε-squares.

Ex: • 1 pt $\quad N(\varepsilon) = 1 = \frac{1}{\varepsilon^{\boxed{0}}}$

• Line: $\quad N(\varepsilon) \propto \frac{L}{\varepsilon^{\boxed{1}}}$

• 2D region: $\quad N(\varepsilon) \propto \frac{H \cdot W}{\varepsilon^{\boxed{2}}}$



Dim

In general a d-dim. region needs $N(\varepsilon) \propto \varepsilon^{-d}$

$$\Rightarrow N(\varepsilon) = k\,\varepsilon^{-d} \Rightarrow \ln N = \ln k - d \ln \varepsilon$$
$$\Rightarrow \underline{\ln N - \ln k} = d \ln 1/\varepsilon$$
$$\Rightarrow d = \frac{\ln N - \ln k}{\ln 1/\varepsilon}$$

Since we want $\varepsilon \to 0$: $\ln k$ is negligible.

def : $\boxed{D_B = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon}}$ Fractal dimension

Also called : • box-counting dim.
• capacity dim.

⚠ ① in practice ε will be small but not → 0
② Boxes can overlap and move
③ Boxes can have any shape.