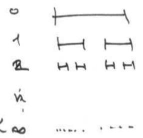
4.6 Computing the box-counting dim.

Ex: Cantor set K:



0: $\varepsilon = 1,\ N = 1$

1: $\varepsilon = 1/3,\ N = 2$

2: $\varepsilon = 1/3^2,\ N = 2^2$

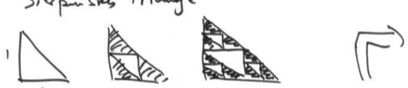
n: $\varepsilon = 1/3^n,\ N = 2^n$

K_∞ ....

$$D_B = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon} = \lim_{\varepsilon \to 0} \frac{\ln 2^n}{\ln 3^n} = \lim_{n \to \infty} \frac{\cancel{n} \ln 2}{\cancel{n} \ln 3}$$

$$D_B = \ln 2 / \ln 3 = 0.6309....$$

Ex: Sierpinski triangle



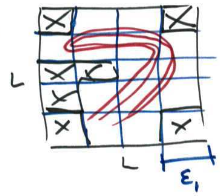
$N=1$, $\varepsilon = 1$ — $N=3$, $\varepsilon = 1/2$ — $N = 3^2$, $\varepsilon = 1/2^2$ ... $N = 3^n$, $\varepsilon = 1/2^n$

$$D_B = \dots = \frac{\ln 3}{\ln 2} = 1.5849\dots$$

Q: $D_B = ?$ If we have an orbit?

Ex: Hénon

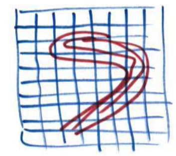
$\varepsilon_1 = \#$



$N_1 = 13$

$\varepsilon_1 = \#$

$\varepsilon_2 = \varepsilon/2$



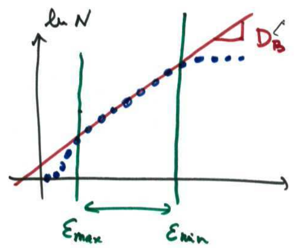
$N_2 = \#$

$\varepsilon_2 = \#$

1) store Many Many Many pts of the object (orbit)
2) set a grid size ε ($=0.1$ for example)
3) count # of boxes that contain @ least 1 pt.
4) $\varepsilon = \varepsilon/2$

$$D_B = \lim_{\varepsilon \to 0} \frac{\ln N}{\ln 1/\varepsilon}$$

plot:



$D_B \approx$ slope of linear region

$\longrightarrow$ $D_B \approx$ 1.216 Carreteo's; 1.27 Book; ___ yours.

Hausdorff dim: textbook: $D_B \geq D_H$

and in most cases $D_B \simeq D_H$

4.7 Correlation Dim. (Grassberger & Procaccia '83)

$\longrightarrow$ this Dim. is best suited for ORBITS.

Def: let $S = \{x_0, x_1, \dots\}$ be an orbit of the map f on $\mathbb{R}^m$. For each $r > 0$ define the correlation function $C(r)$ to be the proportion of pairs of pts in the orbit within r units of one another.

Def: S_N the first N pts of orbit.

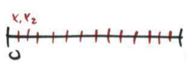
$$C(r) = \lim_{N \to \infty} \frac{\#\{\text{pairs } (x_i, x_j) \text{ such that } x_i, x_j \in S_N : |x_i - x_j| < r\}}{\#\{\text{pairs in } S_N\}}.$$

⚠ By construction $0 \leq C(r) \leq 1$

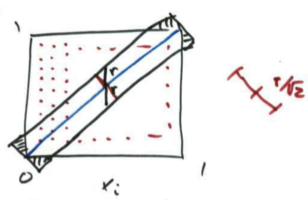
Def: CORRELATION DIM:

$$\boxed{D_c = \lim_{r \to 0} \frac{\ln C(r)}{\ln r}}$$

Ex: 1D segment:



plot all possible (x_i, x_j)

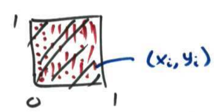


$$C(r) \sim \frac{\text{Area of band}}{\text{Total Area}} = \frac{A_B}{A_T} = \frac{A_B}{1}$$

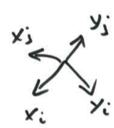
$$A_B = L \times W = \cancel{\sqrt{2}} \times 2 \frac{r}{\cancel{\sqrt{2}}} = 2r$$

$$\therefore\ D_c = \lim_{r \to 0} \frac{\ln C(r)}{\ln(r)} = \lim_{r \to 0} \frac{\ln(2r)}{\ln r} = \lim_{r \to 0} \frac{\ln 2 + \ln r}{\ln r} = 1$$

Ex: 2D area



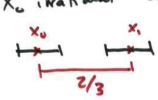
plot in 4D all possible pairs (x_i, y_i, x_j, y_j)



.... $D_B = 2$

Ex: Cantor set: take orbit on Cantor set. (cf. x_0 irrational and $x_{n+1} = T_3(x_n)$)

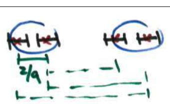
K_1: $n = 1$



$C(r) = \frac{\# < r}{\#}$

$N = 1$

K_2: $n = 2$



$\frac{2}{4 \cdot 3}$

K_3

$N = 2^3$

$\frac{4}{N(N-1)}$

n: $\frac{2^{n-1}}{N(N-1)}$, $r = \frac{2}{3^n}$, $N = 2^n$

$$\therefore\ C(r) = \frac{2^{n-1}}{2^n(2^n - \cancel{1})} \overset{|n| \gg 1}{\approx} \frac{2^{n-1}}{2^{2n}} \qquad \lim_{r \to 0} \longrightarrow \lim_{n \to \infty}$$

$$D_c = \lim_{n \to \infty} \frac{\ln C(r)}{\ln r} = \lim_{n \to \infty} \frac{\ln[2^{n-1-2n}]}{\ln[2/3^n]}$$

$$= \lim_{n \to \infty} \frac{\ln 2^{-n-1}}{\ln 2 - \ln 3^n} = \lim_{n \to \infty} \frac{\ln 2^{-n} + \cancel{\ln 2^{-1}}}{\cancel{\ln 2} - \ln 3^n}$$

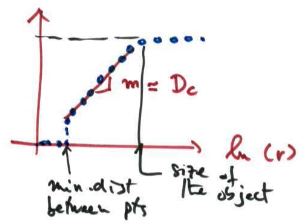
$$= \lim_{n \to \infty} \frac{-\cancel{n} \ln 2}{-\cancel{n} \ln 3} = \frac{\ln 2}{\ln 3} = D_B$$

⚠ In general $D_B \neq D_c$, in fact it can be shown that $D_c \leq D_B$

but in practice $D_B \approx D_c$

- Ex: Hénon:
  - $D_B \simeq 1.27$, $D_c \simeq 1.23$] book
  - $D_B \simeq 1.2183$, $D_c \simeq 1.2126$] Carretero
  - $\boxed{\quad}$] yours.

In practice: $\ln C(r)$



- Other dim:
  - Information dim
  - Renyi dim
  - Hausdorf
  - Lyap. Dim ← Next class