CHap 5: chaos in 2D

1D: Lyap. exp: "easy" because a single dir. for expansion: x-dir.

2D: ≠ exp. along ≠ directions → compute Lyap. exp?

5.1 Lyap. Exp.

For an m-dim. map ($\mathbb{R}^m$) there are m Lyap. exp. measuring expans./contract. along m orthogonal directions.

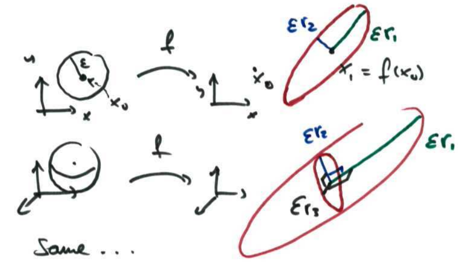
(a) First direction = direction of greatest stretching (or least contracting).

(b) Second dir = direc. of greatest stretch ⊥ to (a)

(c) 3rd dir = dir. of greatest stretch ⊥ to (a) & (b)

In 2D:

In 3D:



In ND: Same ...

Def: 5.1 Let f be a smooth map on $\mathbb{R}^m$, let $J_n = D[f^n(v_0)]$ (= Jacobian of the n-th iterate of map) and $\varepsilon r_k^{(n)}$ the length of the k-th largest ⊥ axis of the ellipsoid $J_n[N_\varepsilon(v_0)]$.
Thus $r_k^{(n)}$ measure the contrac/exp. rates after n iterates about v_0 (I.C.)

then the k-th Lyap. # of v_0 is def.

$$L_k = \lim_{n\to\infty} (r_k^{(n)})^{1/n}$$

is it exists. And Lyap. Exp:

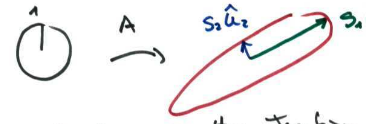
$$\lambda_k = \ln(L_k)$$

- Def: the set $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ is called Lyap spectrum.
- By construction: $\lambda_1 \geq \lambda_2 \geq \lambda_3 \dots \geq \lambda_m$

Sec. 2.7 Matrix * circle = ellipse

Theo 2.24: let N be the unit ~~disk~~ sphere $\mathbb{R}^m$ and let A be an m×m real matrix. Denote $s_1^2, s_2^2, \dots, s_m^2$ and $\hat{u}_1, \dots, \hat{u}_m$ the evals & evecs of AA^T

1. $\hat{u}_i$ are ⊥
2. $A\cdot[N]$ is an ellipsoid with axes $\hat{u}_i$ & lengths s_i



Nonlinear map: use the Jacobian !!!

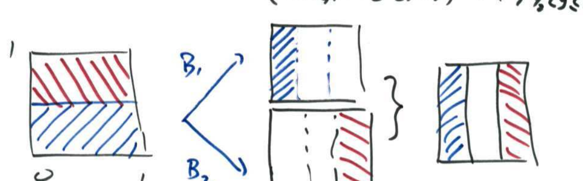
Def: 5.2 CHAOS: Let f be a map on $\mathbb{R}^m$ and let $\{\vec{x}_i\}_{i=0}^{\infty}$ be a bounded orbit of f. The orbit is chaotic if:

1- The orbit is not asymp periodic
2- No Lyap # is exactly = 1 [or Lyap. exp. = 0]
3- $L_1(\vec{x}_0) > 1$ or $\lambda_1(\vec{x}_0) > 0$ [i.e. have @ least 1 dir. of expan]

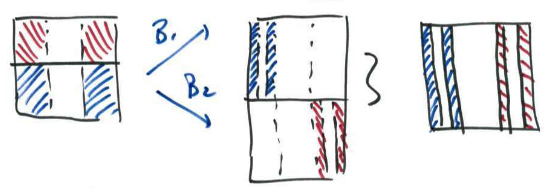
Ex: 5.3 Skinny baker's map

$$\binom{x}{y} \mapsto B\binom{x}{y} = \begin{cases} B_1\binom{x}{y} = \begin{pmatrix} 1/3 & 0 \\ 0 & 2 \end{pmatrix}\binom{x}{y} & \text{if } 0 \le y \le 1/2 \\ B_2\binom{x}{y} = \begin{pmatrix} 1/3 & 0 \\ 0 & 2 \end{pmatrix}\binom{x}{y} + \binom{2/3}{-1} & \text{if } 1/2 < y \le 1 \end{cases}$$

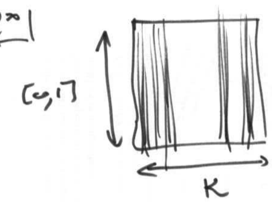
f1



f2



f∞



Cantor set of lines.

Chaos: + Jacobian if $y \neq 1/2$ → $J = \begin{pmatrix} 1/3 & 0 \\ 0 & 2 \end{pmatrix}$

+ J is symmetric ∴ $J = J^T$

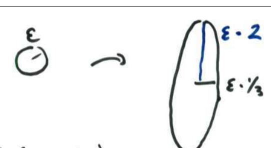
⟹ • $\text{Eig}(JJ^T) = \text{Eig}(JJ) = \text{Eig}(J^2) = [\text{Eig}(J)]^2$

• $\text{Evecs}(A^n) = \text{Evecs}(A)$

∴ I can use evals, evecs of J.

$$\begin{cases} \mu_1 = 2, & \vec{v}_1 = \binom{0}{1} \\ \mu_2 = 1/3, & \vec{v}_2 = \binom{1}{0} \end{cases}$$

In 1 iterate



Since J = const (if $y \neq 1/2$)

$$\Rightarrow L_k = \lim_{n\to\infty} (r_k^{(n)})^{1/n} = \begin{cases} 2 & k=1 \\ 1/3 & k=2 \end{cases}$$

$$\therefore \quad \lambda_1 = \ln(2) = 0.6913 > 0$$
$$\lambda_2 = \ln(1/3) = -1.0986 < 0$$

∴ ALL orbits that are not period will be chaotic.

Ex: the Arnold's CAT map

$$\binom{x}{y} \mapsto f\binom{x}{y} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\binom{x}{y} \pmod 1$$

Away from disc. $J = \text{const} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

J is symm. ⟹ Eig(J).

$$\mu_1 = \frac{3+\sqrt{5}}{2} = 2.618$$

$$\mu_2 = \frac{3-\sqrt{5}}{2} = 0.382$$

$$\Rightarrow \lambda_1 = \ln(\mu_1) = 0.962\ldots > 0$$
$$\lambda_2 = \ln(\mu_2) = -0.962\ldots < 0$$

∴ Any orbit NOT event. periodic will be CHAOTIC

Area preservation: Along orbit:
$$\begin{cases} \Sigma\lambda_i < 0 \to \text{area contraction} \\ \Sigma\lambda_i = 0 \to \text{area preserv.} \\ \Sigma\lambda_i > 0 \to \text{area expansion.} \end{cases}$$