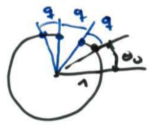
20.1

Ex 5.5 : $f(r,\theta) = (r^2, \theta + q)$

• If $r_0 = 1 \to$ stay on circle



$$Df = \begin{pmatrix} 2r & 0 \\ 0 & 1 \end{pmatrix}$$

• If $r_0 = 1 \Rightarrow r = 1 \Rightarrow Df = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$\mu_1 = 2, \ \mu_2 = 1$

J const and symm. $\Rightarrow \begin{cases} \lambda_1 = \ln(\mu_1) = \ln 2 \\ \lambda_2 = \ln(\mu_2) = \ln 1 = 0 \end{cases}$

take $q \in \mathbb{I} \to$ orbit is No asymp. periodic:

Conditions for chaos:
1 = No asymp. periodic ✓
3 = $\lambda_1 > 0$, $\lambda_1 = \ln 2 > 0$ ✓
2 = No $\lambda_i = 0$ ✗

20.2

5.2 Numerical computation of Lyap. Exp.

$$10^{-16} \cdot (r)^p \sim 1$$

ln: $-16 \ln 10 + p \ln r \sim 0$

$$\Rightarrow \quad p \approx \frac{16 \ln 10}{\ln r} = \frac{16 \ln 10}{\ln(1.01)} \simeq 3K$$

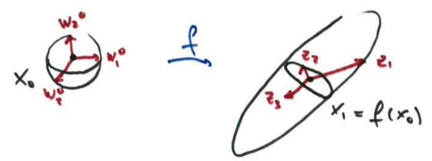
Computing Lyap. Exp. the DIRECT way IS IMPOSSIBLE.

DIRECT:
• take $p \gg 1$
• compute Df^p
• do evals $((Df^p)(Df^p)^T)$

∴ We need a practical method.

→ Indirect approach.

20.3



• Take ⊥ basis on ε-ball: $\{w_i^0\}$ at x_0.

• $z_i = f(w_i^0)$

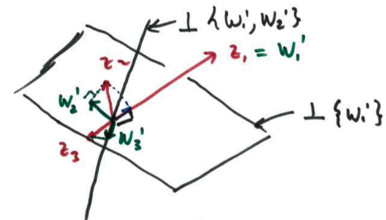
⚠ $\{z_i\}$ is Not ⊥ and it is not along axis of ellipsoid :(

A: apply Graham-Schmidt orthogonalization

→ construct ⊥ basis from $\{z_i\}$.

∘ $w_1^1 = z_1$ (the LARGEST) ∘ etc
∘ w_2^1 = project of z_2 ⊥ to z_1 ∘ etc
∘ w_3^1 = project of z_3 ⊥ $\{z_1, w_2^1\}$ ⋮

20.4



Math:

$$w_1' = z_1$$

$$w_2' = z_2 - \frac{(z_2 \cdot w_1') \cdot w_1'}{\|w_1'\|^2} \quad \text{Norm 1}$$

$$w_3' = z_3 - \frac{z_3 \cdot w_1'}{\|w_1'\|^2} w_1' - \frac{z_3 \cdot w_2'}{\|w_2'\|^2} w_2'$$

⋮

$\{w_i^1\}$ is ⊥

20.5

Do it AGAIN: apply f and do G-S ⊥.

• $z_i = f(w_i^1)$
• w_i^2 using GS on the $\{z_i\}$

∘ ∘ ∘ Many times.

Problem: w_1^p will be MUCH larger that $w_2^p \Longrightarrow$ when projecting → LARGE ERRORS !! :(

A: Normalize @ each iteration (each vector individually)

→ these normalization scalars are the Lyap #'s !!! :)

20.6

@ each iteration: Normalize:

$$\overline{w_1'} = \varepsilon \frac{w_1'}{\|w_1'\|}, \qquad \overline{w_2'} = \varepsilon \frac{w_2'}{\|w_2'\|}, \quad \vdots$$

$\{\overline{w_i'}\}$ is ⊥ and NORMALIZED to size ε.

expansions along the ≠ dir. of the ellipsoid.

Along the k-direction we apply time "average"

$r_k^{(n)} \approx \|w_k^1\| \cdots \|w_k^n\|$ after n-it.

$$\Rightarrow L_k = \lim_{n \to \infty} \left(r_k^{(n)} \right)^{1/n}$$

$$\Rightarrow \boxed{\lambda_k = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln \|w_k^i\|}$$

20.7

HW: find Lyap. Exp. for

* Hénon: $\lambda_1 = 0.42$, $\lambda_2 = -1.62$
* Ikeda: $\lambda_1 \simeq 0.51$, $\lambda_2 \simeq -0.72$

5.3 Lyap. Dim. $\boxed{D_L}$

D_B: box-counting

D_C: correlation-dim.

D_L:
• No boxes needed :)
• GS needs implementing :(

Def 5.8. • f be a map $\mathbb{R}^m$, consider orbit with Lyap Exp:

$\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ ← Lyap. SPECTRUM

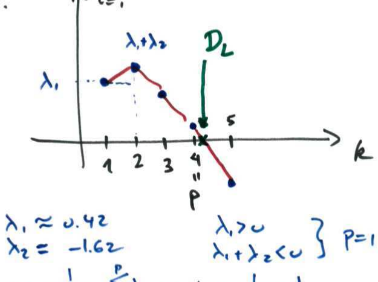
• Denote p the largest integer such that $\sum_{i=1}^{p} \lambda_i \geq 0$

20.8

• Def Lyap. dim. D_L:

$$D_L = \begin{cases} 0 & \text{if no such } p \text{ exists } (\lambda_1 < 0) \\ p + \dfrac{1}{|\lambda_{p+1}|} \sum_{i=1}^{p} \lambda_i & \text{if } p < m \\ m & \text{if } p = m \end{cases}$$

Graphically:



Ex: Hénon: $\lambda_1 \simeq 0.42$, $\lambda_2 = -1.62$; $\lambda_1 > 0$, $\lambda_1 + \lambda_2 < 0$ } $p = 1$

$$D_L = p + \frac{1}{|\lambda_{p+1}|} \sum^{p} \lambda = 1 + \frac{1}{|\lambda_2|} \lambda_1$$

20.9

$$\Rightarrow D_2 = 1 + \frac{0.9^2}{1.62} = 1.26$$

$$(D_B = 1.27,\ D_c = 1.23)$$

<u>Ex</u>: Skinny Baker's map : $\lambda_1 = \ln 2 = 0.69$
$\lambda_2 = \ln 1/3 = -1.09$

$$p=1 \Rightarrow D_2 = 1 + \frac{\lambda_1}{|\lambda_2|} = 1 + \frac{\ln 2}{|\ln 1/3|}$$

$$= 1 + \frac{\ln 2}{|-\ln 3|} = \boxed{1 + \frac{\ln 2}{\ln 3} = D_2}$$

$$\underbrace{\begin{matrix} D_B(K) = \frac{\ln 2}{\ln 3} \\ D_B(1D) = 1 \end{matrix}}$$

$$D_B[\text{Baker's}] = 1 + \frac{\ln 2}{\ln 3}$$

$=$