21.1

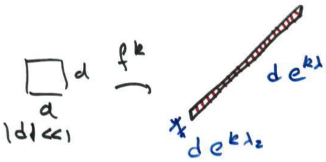
Relation between ≠ dimensions

D_I : inf. dim.
D_H : Hausdorff dim.

Prop :
- $D_H \leq D_B$
- $D_C \leq D_I = D_L \leq D_B$
- ~~Kaplan-Yorke Conjecture~~: $D_L = D_B$
- $D_L = D_I \leq D_B$: $D_L \leq D_B$

KY conj :

2D :



$\Rightarrow$ side of box $\equiv \varepsilon = d e^{k\lambda_2}$

21.2

$$\Rightarrow N(\varepsilon) = \frac{\cancel{d} e^{k\lambda_1}}{\cancel{d} e^{k\lambda_2}} = e^{k(\lambda_1 - \lambda_2)}$$

$$\therefore D_B = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{-\ln \varepsilon} = \lim_{k \to \infty} \frac{k(\lambda_1 - \lambda_2)}{-[\cancel{\ln d} + k\lambda_2]}$$

$$= \frac{\lambda_1 - \lambda_2}{-\lambda_2} = 1 - \frac{\lambda_1}{\lambda_2} = 1 + \frac{\lambda_1}{-\lambda_2}$$

$$\boxed{D_B = 1 + \frac{\lambda_1}{|\lambda_2|}} \boxed{= D_L}$$

& something similar can be done in N-D.

⚠ This is FALSE $D_L \leq D_B$

Hénon :

	D_B	D_C
Cantor	1.2143	1.2126
Jenny	1.2266	1.2007
Katie	1.1832	1.2103
Greg	1.2106	1.2103
John	1.2293	1.1848
Chris	1.2199	1.2187

$D_C \leq D_B$

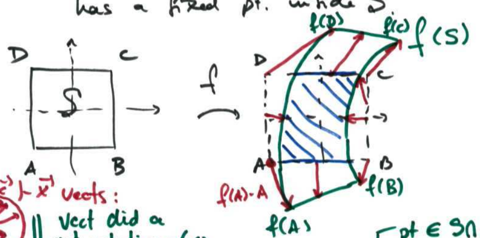
21.3

5.4 Fixed pt. theo. in 2D

Theo 5.10 : Let f be a cont. map on $\mathbb{R}^2$ and S a rect. region such that as the boundary of S is traversed the "net rotation" of the vectors $f(\vec{x}) - \vec{x}$ is non-zero $\Longrightarrow f$ has a fixed pt. inside S.

$\vec{AB}$

Ex : ⓐ



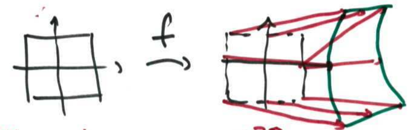
plot all $f(\vec{x}) - \vec{x}$ vects :

|| Vect did a net rotation $\neq 0$
$\xrightarrow{5.10} \exists$ f. pt. in S !!!

[pt $\in S \cap f(S)$

21.4

Ex ⓑ :



plot all vects :

: zero net-rot
∴ No f.p. in S !!!

⚠ • There is nothing specific about S
• S could be ANY region without holes
• ANY shape

Sub-corollary : Brouwer fixed pt. theo.

If f is 1-to-1 , S = square, $f(S) \subset S$
$\Rightarrow \exists$ f. pt. in $f(S)$

⚠ S not need to be a rectangle
$\Rightarrow$ ANY shape works

$\longrightarrow f(S) \subset S \Rightarrow S$ is a trapping region.

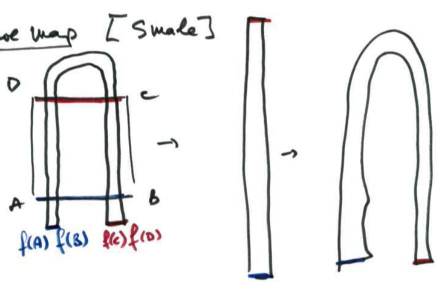
21.5

5.5 Markov partitions ← Not covered
→ see book.

5.6 The Horseshoe map

Essence of chaos : stretching + folding.

· Horseshoe map [Smale]

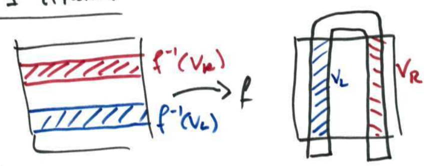


21.6

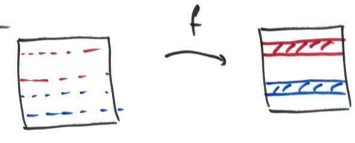
Ideal : inside $W = (A, B, C, D)$ we have const.
{ Y : stretch factor 3
{ X : contract factor 1/4

Follow all pts that stay inside W both in forward and backward time :

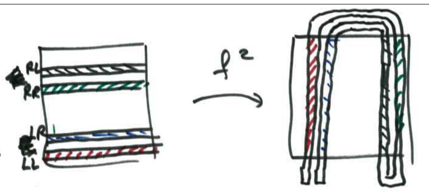
After 1 iteration



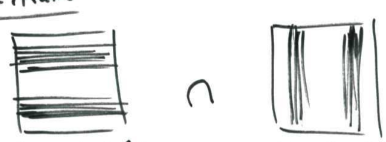
2 iterations

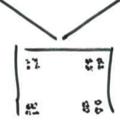


21.7



After n-iterates





← Cantor "DUST"

21.8

Every pt. in Cantor dust will have a UNIQUE symb. dyn. description by a binary DOUBLY infinite seq :

$$\ldots \underbrace{S_{-3} S_{-2} S_{-1}}_{f^{-1}} \underbrace{S_0 \bullet S_1 S_2 \ldots}_{f} \qquad S_i = \begin{cases} R \\ L \end{cases}$$

And iterating through f : displacing "•" to right
——— " ——— f^{-1} : ——— " ——— " ——— to left.

$$f(\ldots S_{-1} S_0 \bullet S_1 S_2 \ldots) = S_{-1} S_0 S_1 \bullet S_2$$

SHIFT map.

in Y : 3×
in X : 1/4×
} $\lambda_1 = \ln 3 > 0$
$\lambda_2 = -\ln 4$

$\Longrightarrow$ CHAOS if we ~~see~~ start inside Cantor dust.