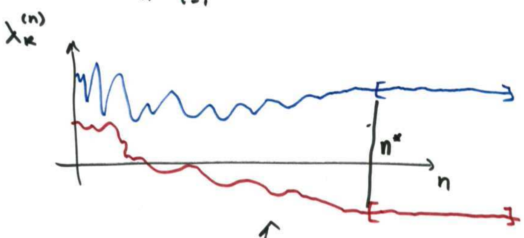
Convergence of Lyap. Exp.

$$\lambda_k^{(n)} = \frac{1}{n}\sum_{i=1}^{n} \ln |r_k^{(i)}|$$



HW: ① Include these
② Include code.

Chap 6. chaotic attractors

chaotic attractor: 1) Contain a chaotic orbit
2) that attracts a non-zero area of pts.

6.1 Forward limit sets

Ex: Comp. Exp. 6.2

Def 6.1: Let f be a map and x_0 be a IC. The forward limit set of the orbit $\{f^n(x_0)\}_{n=0}^{\infty}$ is the set

$$\omega(x_0) = \{ x \mid \forall N, \varepsilon > 0 \ \exists\ n > N \text{ such that } |f^n(x_0) - x| < \varepsilon \}$$

Def: $\omega(x_0)$ = omega-limit set.
↑ last letter of Greek alphabet.

Prop: * the orbit itself may have NO pts on its ω-limit set.
ex: orbit approaching a sink
ω = fixed pt.

* ω(periodic orbit) = periodic orbit
* ω-limit set may be the whole domain.
ex: logistic map: a chaotic orbit will fill $[0,1]$ densely
$\Rightarrow \omega(x_0) = [0,1]$
← chaotic orbit

Def: 6.2: * Let $\{f^n(x_0)\}$ be a chaotic orbit. If $x_0 \in \omega(x_0)$ then $\omega(x_0)$ is called a chaotic set.

* An attractor is an ω-limit set that attracts a non-zero measure set.
* A chaotic attractor = chaotic set + attractor.

⚠ chaotic set $\not\Rightarrow$ chaotic attractor
ex: $T_3(x)$

Any $x_0 \in K \to$ chaotic set
but it is NOT attracting. :(

⚠ bif. diag.

⚠ • It is almost always impossible to prove analytically that a given orbit is a chaotic attractor ... we try our best numerically!

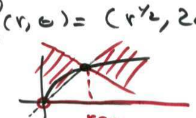
6.2 chaotic attractors

- Sink (f.pts or periodic orbits) can ~~not~~ easily verify the condition of "attractiveness"
- chaotic attractor: ?

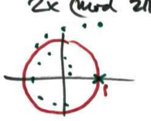
ex: $f(x) = 2x \pmod 1$
$\to x_0$ is irrational $\to f^n(x_0)$ is dense
& $\lambda = \ln 2 \Rightarrow \omega(x_0)$ is a chaotic attractor.

ex: $\boxed{2D}$ $f(r,\theta) = (r^{1/2}, 2\theta)$

$\boxed{r}$:



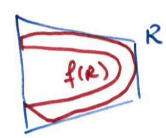
$\boxed{\theta}$: $2x \pmod{2\pi}$



$\omega(x_0) = \odot$
← not per.

Ex: Hénon map

$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a - x^2 + by\\ x\end{pmatrix}$$



$f(R) \subseteq R$
$\Rightarrow f$ has a TRAPPING region.

- Area: $|J|$ gives area contract/exp.

$$J = \begin{pmatrix}-2x & b\\ 1 & 0\end{pmatrix}$$

$\Rightarrow |J| = |b|$

$\therefore a = 1.4,\ b = 0.3 \Rightarrow |J| = 0.3$

$\therefore$ After ∞-# of iterations
$A(f^{\infty}(R)) = 0$

- HW: $\lambda_1 \approx 1.6\ldots > 0$

$\Rightarrow$ chaotic attractor.

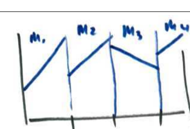
Main issue to prove any analytical result: infinitesimal changes on a & b can lead to "structural" changes of the attractor.

chaotic att $\xrightarrow[\text{in a or b}]{\text{change}}$ fix pt. / periodic orbit.

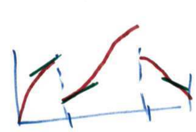
6.3 chaotic attractors of expanding maps

Def: a map f is piece-wise expanding with stretching factor α if for every piece $[P_{i-1}, P_i]$ $|f'(x)| \geq \alpha > 1$ on the stretching partition $\{P_0, \ldots, P_k\}$ with $P_{i-1} < P_i$.

ex:

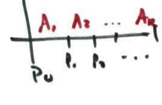


$\min_i \{|M_i|\} = \alpha$
If $\alpha > 1 \Rightarrow$ stretching.



$\min\{|M|\} = \alpha$
If $\alpha > 1 \Rightarrow$ stretching.

Consequences: If one follows the symbolic dynamics of an orbit:



L = length(I)
$I = \cup A_i$

(except countably many orbits that fall on the P_i's)

We have that for every <u>allowable</u> sequence there will be a subinterval of length at most L/α^{n-1}, called an n-subinterval, whose pts follow this seq.

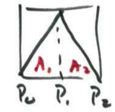
∴ an infinite allowable seq. will represent a single pt in I since $\frac{L}{\alpha^{n-1}} \xrightarrow{n\to\infty} 0$

* Furthermore: $(f^n)'(x_0) = f'(x_{n-1}) \cdots f'(x_0) \geq \alpha^n$

$$\Rightarrow \lambda(x_0) \geq \ln \alpha \underset{\alpha>1}{>} 0$$

∴ Any seq. that is not periodic or event. per. will be a CHAOTIC orbit.

ex: tent map



$\alpha = 2 > 1$

∴ $\{0, 1/2, 1\}$ is a stretching partition and $\alpha = 2$ ∴ $\lambda \geq \ln 2$

ex: W-map: