Theo: 1) Each allowable seq. $A_1 \dots A_n$ corresponds to a sub-interval of length @ most L/α^{n-1}

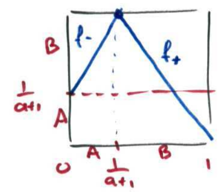
2) Each allowable infinite seq. corresponds to a single pt. in I and if seq. is not event. periodic it is chaotic ($\alpha > 1$)

3) If each pair of symbols, B and C [C might be symbol B], can be connected by a finite seq.

B ..... C and C ..... B

then f has a dense orbit on I and I is a chaotic attractor.

Ex: $f(x) = \begin{cases} \frac{1}{a+1} + ax = f_-(x) & 0 \le x \le 1/a+1 \\ a(1-x) = f_+(x) & 1/a+1 < x \le 1 \end{cases}$ with $a = \frac{\sqrt{5}+1}{2} \approx 1.618$



- Not every seq. is allowed
- But every connection is possible.

AA : A → B → A ✓

AB : A → B

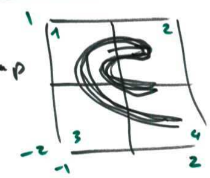
BA : B → A ✓

BB : B → B ✓

∴ theo → chaotic orbit ($\alpha > 1$, $\lambda = \ln 1.618 > 0$) that densely covers $I = [0,1]$.

6.4 Measure

Ex: Ikeda map



1 30%

2 36%

3 17%

4 17%

* The formal concept of histogram will come from "measure"

Properties: (a) The measure of any set is non-negative $\mu(A) \geq 0$

(b) The measure of ~~two~~ union of distinct sets is the sum of their individual measures.

$\mu(A \cup B) = \mu(A) + \mu(B)$ if $A \cap B = \phi$

Any function of sets in $\mathbb{R}^n$ satisfying (a) & (b) is called a MEASURE.

Ex: ordinary measures:

$\left.\begin{array}{l} \mathbb{R}^1 : \text{length} \\ \mathbb{R}^2 : \text{area} \\ \mathbb{R}^3 : \text{volume} \end{array}\right\}$ Lebesgue measures

More properties

(c) The measure of the whole space = 1

→ Measure is called probability measure.

(d) $\mu(f^{-1}(S)) = \mu(S)$ for any closed set S

→ Measure is called INVARIANT measure.

6.5 Natural measure : (this defines histograms)

First def. the fraction of iterates of the orbit $\{f^n(x_0)\}$ lying in a set S.

$$F(x_0, S) = \lim_{n\to\infty} \frac{\#\{f^k(x_0) \in S \text{ for } 1 \le k \le n\}}{n}$$

Ex: Take $f(x) = x/2$

- If S is an open set containing 0 $\Rightarrow F(x_0, S) = 1$
- $F(x_0, [0,\infty)) = \begin{cases} 0 & \text{if } x_0 < 0 \\ 1 & \text{if } x_0 \geq 1 \end{cases}$

In order to avoid checking on which side → replace S by an Epsilon-Neig. of S.

$F(X_0, \underbrace{N(\varepsilon, S)}_{\hookrightarrow \text{ eps. neig. for sets.}})$

$N(\varepsilon, S) = \{x \text{ such that } \operatorname{dist}(x, S) \le \varepsilon\}$

Def 6.14: The Natural measure generated by the map f is:

$$\mu_f(S) = \lim_{\varepsilon \to 0} F(x_0, N(\varepsilon, S))$$

for a given closed set S as long as all x_0,

with exception of a set of measure 0, give the same answer.

Ex: $f(x) = x/2$

- $\mu_f(0) = 1$
- $\mu_f(S) = \begin{cases} 1 & \text{if } 0 \in S \\ 0 & \text{if } 0 \notin S \end{cases}$

6.6 Invariant measure for 1D maps

→ How to build ~~the~~ μ_f in practice.

Let us express measure using integrals: for a μ: find $p(x)$ such that:

$$\mu(S) = \int_S p(x)\,dx$$

$p(x)$ is called the density of the measure μ.

Ex: W-map

... density $p(x) = \begin{cases} 0 & 0 \le x < 1/4 \\ 2 & 1/4 \le x < 1/2 \\ 2 & 1/2 \le x < 3/4 \\ 0 & 3/4 < x \le 1 \end{cases}$

p(x):

- $\int_0^1 p(x)\,dx = 1$ ∴ P is a density of a probability measure
- Invariance : Any set S: $\mu(f^{-1}(S)) = \mu(S)$

Ex: • $S = [\frac{1}{4}, \frac{3}{4}] \Rightarrow \mu(S) = \int_S p(x)\,dx$

$= \int_{1/4}^{3/4} 2\,dx = [2x]_{1/4}^{3/4}$

$= 2(3/4 - 1/4) = 1$

• $S = \cancel{[\frac{1}{4}, \frac{1}{2}]}$ $S^{-1} = f^{-1}(S)$

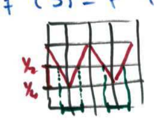
$[0,1] = f^{-1}([1/4, 3/4])$ =

$\mu(f^{-1}(S)) = \int_0^1 p(x)\,dx = 1$

• $S = [\frac{1}{4}, \frac{1}{2}] \Rightarrow \mu(S) = \int_{1/4}^{1/2} p(x)\,dx$

$= [2x]_{1/4}^{1/2} = 2 \cdot 1/4 = 1/2$

$S^{-1} = f^{-1}(S) = f^{-1}([1/4, 1/2]) = [\frac{1}{8}, \frac{3}{8}] \cup [\frac{5}{8}, \frac{7}{8}]$

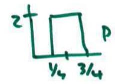


$\mu(f^{-1}(S)) = \int_{\frac{1}{8}}^{3/8} p(x)\,dx + \int_{5/8}^{7/8} p(x)\,dx$

$= 2 \times \frac{1}{8} + 2 \times \frac{1}{8}$

$= 1/2 = \mu(S)$

∴

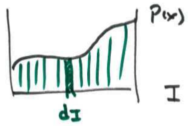


p(x) is an invariant probability measure for W-map !!!

Def: If p(x) generates a prob. measure μ then p(x) is called probability density function (PDF) for μ.

Main Result: The prob. dens. funct. p(x) is the Normalized ($\int p = 1$) histogram corresponding to generic ICs and $\mu(S)$ [$= \int_S p(x)dx$] gives the fraction of ~~pts~~ how often we hit a pt. in S.

$\Rightarrow$ $\mu(S) \cdot dS$ gives the fraction of pts inside dS. 23.9



$\int p(x) = 1$

prob : $p(x) \cdot dI$