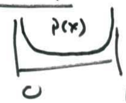
24.1

μ for Logistic map

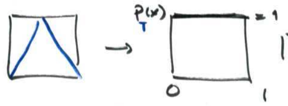
- hist:



$\therefore \quad \mu(S) = \int_S p(x)\,dx$

- use conjugacy $T \longleftrightarrow$ Log. Map

T:



$\boxed{P_T(x) = 1}$

Conj:

$$\begin{array}{ccc} x & \xrightarrow{T} & T(x) \\ C\downarrow & & \downarrow C \\ y & \xrightarrow{G} & * \end{array} \qquad \begin{array}{l} C(T(x)) = G(C(x)) \\ T^{-1}(S) = C^{-1}G^{-1}C(S) \end{array}$$

- $\mu_T(S) = \int_S P_T(x)\,dx$ is an inv. prob measure for T

$$\mu_T(S) = \mu_T(T^{-1}(S))$$

24.2

$$\mu_T(S) = \mu_T(C^{-1}G^{-1}C(S))$$

def: $\mu(S) \equiv \mu_T(C^{-1}(S))$ ← RHS

$[\mu(C(S)) = \mu_T(S)]$ ← LHS

$$\Rightarrow \underbrace{\mu(C(S))}_{LHS} = \underbrace{\mu_T(C^{-1}[G^{-1}(C(S))])}_{RHS}$$

$$\mu(C(S)) = \mu(G^{-1}(C(S)))$$

def: $S' = C(S)$

$\Rightarrow \boxed{\mu(S') = \mu(G^{-1}(S'))}$ $\therefore$ μ is an invariant measure for G.

G: $p = ?$

$$\mu(S) = \mu_T(C^{-1}(S)) = \int_{x \in C^{-1}(S)} P_T\,dx = \int_{x \in C^{-1}(S)} 1\,dx$$

Change of variables: $y = C(x) \Rightarrow x = C^{-1}(y)$

$\Rightarrow dy = C'(x)\,dx \Rightarrow dx = \dfrac{dy}{C'(x)}$

$\Rightarrow dx = \dfrac{dy}{C'(C^{-1}(y))}$

24.3

$$\mu(S) = \int_{\substack{y \in C^{-1}(C(S)) \\ y \in S}} \frac{dy}{C'(C^{-1}(y))} = \int_S \frac{dy}{C'(C^{-1}(y))} \quad (*)$$

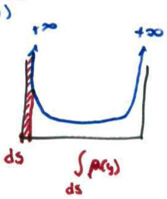
$y = C(x) = \dfrac{1 - \cos \pi x}{2}$, $\quad C'(x) = \dfrac{\pi \sin \pi x}{2}$

$C^{-1}(y) = \dfrac{\cos^{-1}(1-2y)}{\pi}$

$\Rightarrow C'(C^{-1}(y)) = \dfrac{\pi \sin \cancel{\pi}\left(\cos^{-1}(1-2y)/\cancel{\pi}\right)}{2}$

$\cdots = \pi\sqrt{y(1-y)}$

$(*) \Rightarrow \mu(S) = \int_S \underbrace{\frac{1}{\pi\sqrt{y(1-y)}}}_{p(y)}\,dy$



24.4

Time averages & "ergodicity"

Suppose we take an observable φ from a dyn. syst. f and dyn. syst. lives in a space M

dyn. syst: $f: M \to M$ (M: attractor)

observable: $\varphi: M \to \mathbb{R}$

Time average of φ:

$$\bar{\varphi}_n = \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x_0))$$

Ex: Lyap. exp. $\quad \varphi(x_i) = \ln|f'(x_i)|$

Theo: Birkhoff Ergodic Thm:

If μ is an invariant prob. measure with pdf $p(x)$ then,

a) $\bar{\varphi}_n(x_0)$ converges for almost every x_0

24.5

If additionally: μ is ERGODIC then:

b) $\bar{\varphi}_\infty(x_0) = \int_A \varphi(x)p(x)\,dx$

$\therefore$ temporal averages $\longrightarrow$ spacial averages

Def: Ergodic: A measure μ is ergodic if it is "indecomposable"

Def: indecomposable:

For any invariant set $A \Rightarrow \mu(A) = 0$ or $\mu(A) = 1$

Application: Lyap. Exp.

Ex: Logistic map: $f(x) = 4x(1-x)$, $p(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$

$$\lambda(x_1) = \lim_{n\to\infty} \frac{1}{n}\sum \ln|f'(x_n)|$$

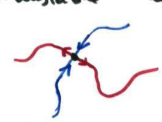
Birkhoff $= \int_0^1 p(x)\ln|f'(x)|\,dx = \int_0^1 \dfrac{\ln|4(1-2x)|}{\pi\sqrt{x(1-x)}}\,dx$

$= 2\int_0^{1/2} \dfrac{\ln 4(1-2x)}{\pi\sqrt{\ }}\,dx = \cdots = \ln 2$ ✓

24.6

CHAP 10: Stable/unstable manifolds (& crises)

Remember



10.1 stable manifold theo

- For linear maps: $W^s \leftrightarrow$ stable evecs, $W^u \leftrightarrow$ unstable evec.
- For nonlinear map: W^s = pts attracted to f. pt. W^u = pts attracted to f. pt using f^{-1}

Theo 10.11: let f be a diffeomorphism (1-to-1, cont. inv. + diff.) of $\mathbb{R}^2$. Assume that f has a saddle f. pt. @ p with $|\lambda_u| > 1$ & $|\lambda_s| < 1$

Proof: 10.4

- The stable W^s and unstable W^u are ONE DIMENSIONAL.
- W^s & W^u are tangential to v^s & v^u (the stable & unstable evecs).

24.7

10.2 Homoclinic & heteroclinic pts

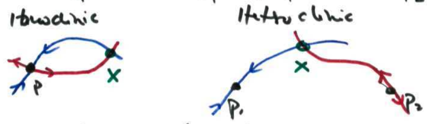
Def: • Let f be an invertible map of $\mathbb{R}^n$ and p be a saddle.

* if $x \in W^s \cap W^u$ & $x \neq p \Rightarrow x$ is a homoclinic pt. and $f^{+\infty}(x) \to p$ & $f^{-\infty}(x) \to p$

* P_1 & P_2: two saddles: $W^s(P_1)$, $W^s(P_2)$, $W^u(P_1)$, $W^u(P_2)$

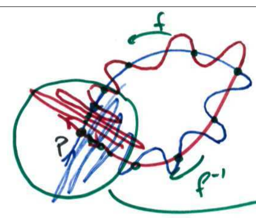
If $x \in W^s(P_1) \cap W^u(P_2) \Rightarrow x$ is a heteroclinic pt. and $f^{+\infty}(x) \to P_1$ & $f^{-\infty}(x) \to P_2$

Homoclinic Heteroclinic



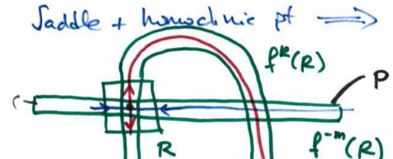
Consequence of finding a homoclinic pt:

24.8



Homoclinic TANGLE.

Saddle + homoclinic pt $\Longrightarrow$ Smale's Horseshoe!



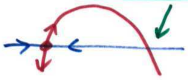
$F = f^{m+k}$

P & $F(P)$

Theo 10.7 [Smale]: Let f be a diffeomorphism in $\mathbb{R}^2$, p is a saddle f.p. If W^s & W^u cross transversally then there is a "hyperbolic" (saddle) horseshoe for some iterate of the map f.

24.9

transversal cross



tangential cross