Mathematical Modelling, Vol. 4, pp. 9–25, 1983
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0270-0255/83 $3.00 + .00

CHAOS IN THE CUBIC MAPPING

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Communicated by Simon A. Levin

(*Received December 1981*)

Abstract—Motivated by a problem in genetics involving one locus with two alleles, R. M. May gave the example of the family of cubic maps $x \to ax^3 + (1 - a)x$ of the interval $[-1, 1]$ and established the existence of a bifurcating sequence of cycles of period 2^n over an interval of parameter values. This paper extends the analysis of May beyond that region. We find a critical value a^* beyond which the map has a snapback repeller and hence chaotic behavior; this value further marks the onset of the first cycle of odd order > 1. When $a = 4$ the map is onto the interval and we find an associated invariant measure.

1. BACKGROUND

In this paper, we study the dynamics associated with a smooth map of an interval having two critical points. The motivation for the problem is provided by a preliminary analysis by May [1, 2] of a simple model in genetics involving one locus and two alleles A and a. Following May we refer the reader to Clarke [3], Wright [4], and Endler [5] for more biological details. The model in question addresses the case where the selective forces depend on the gene frequences in such a way that the allele A has a selective advantage when rare and a disadvantage when common. In these circumstances the gene frequency p of A tends to increase when low and decrease when high, and, with the simplifying change of variable $x = 2p - 1$, the map $f: x_n \to x_{n+1}$ which relates the frequencies in successive generations takes the form shown in Fig. 1. As a specific example, May proposes the cubic map

$$f(x) = ax^3 + (1 - a)x,$$

which is one of the simplest polynomial maps of the desired type. If a is restricted to the range $0 < a \leq 4$ then f maps the interval $I = [-1, 1]$ into itself and we shall study the family $f = f_a$ for these parameter values. For an introduction to the dynamics of maps of the interval see Li and Yorke [6], May [7], May and Oster [8], and the monographs by Collet and Eckmann [9] and Gumowski and Mira [10].

In the remainder of this section, we briefly describe the standard nomenclature and stability criteria for maps of the interval. In Sec. 2, we summarize May's analysis of the map for

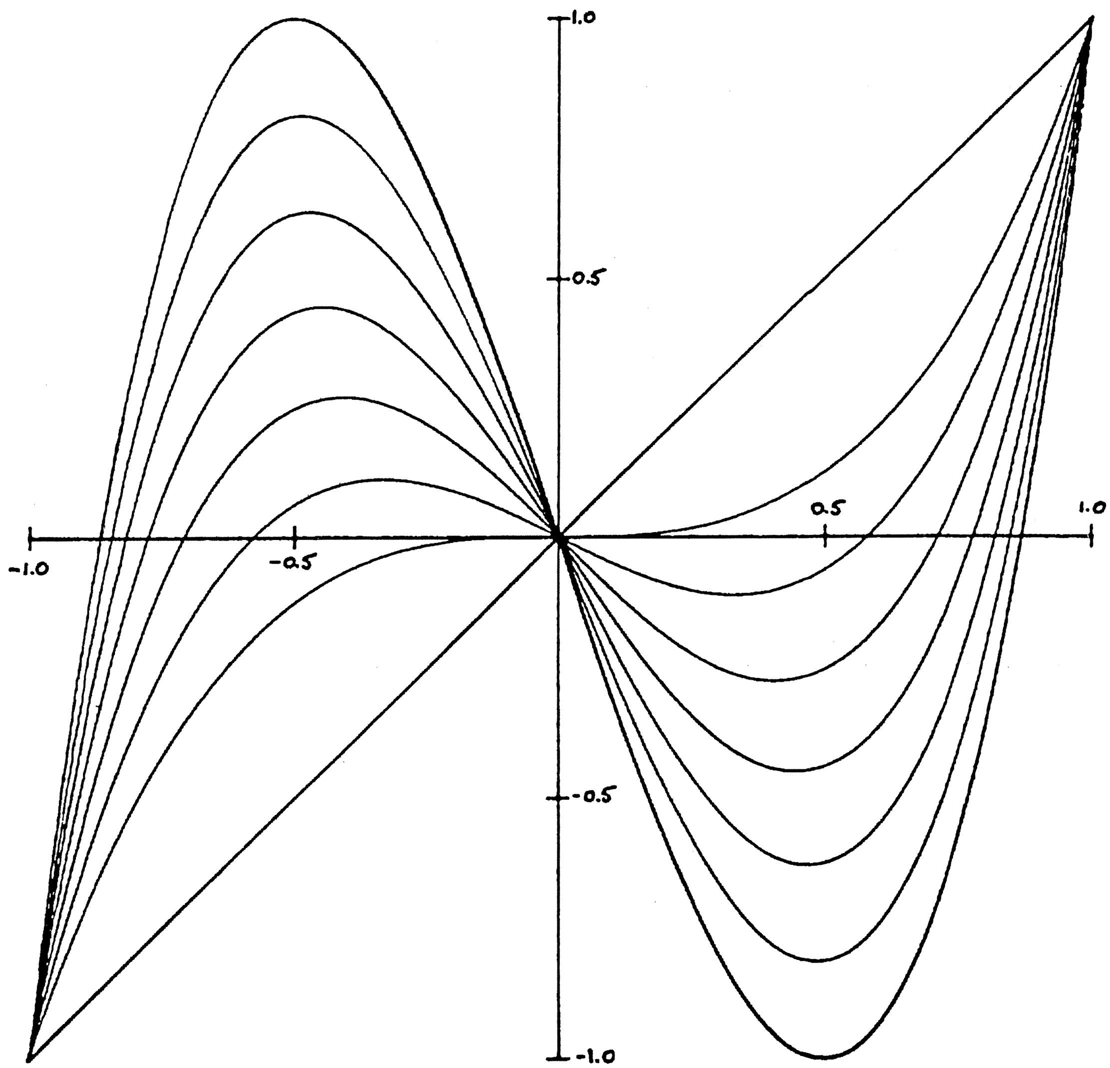

Fig. 1. Graph f_a; $a = 1, 1.5, 2, 2.5, 3, 3.5, 4$. The critical values increases with a.

relatively low values of the parameter and in the third section we describe a computer-drawn bifurcation diagram for the family. Section 4 contains a proof of the existence of chaotic solutions of $x_{n+1} = f_a(x_n)$, using the Li–Yorke definition of chaos, for parameter values larger than a certain $a = a^*$. This result is an application of the snapback repeller theorem of Marotto [11] which provides sufficient conditions for chaotic behaviour in noninvertible maps from $\mathbf{R}^n$ to $\mathbf{R}^n$. We conclude with a description of the dynamics of f_a for two special parameters values: $a = a^*$ and $a = 4$. In both cases, f maps an interval onto itself, and the number of periodic points of f can be counted explicitly. When $a = 4$, f is, in fact, the third Chebyshev polynomial which is topologically equivalent to a piecewise linear map and has a known invariant measure [12]. The map is mixing, and hence ergodic, for this value of a. We thank Phil Holmes for this reference.

We will use the following standard definitions and notation. If $f: I \to I$ is a map of an interval $I \subseteq \mathbf{R}$, let f^n denote the n-fold self-composition of f:

$$f^0 = \text{identity}, \; f^1 = f, \quad f^n = f(f^{n-1}), \quad n = 2, 3 \ldots$$

The *orbit of* x is the set $\{f^n(x)\}_{n=0}^{\infty}$ and the ω*-limit set* of x is the set of limit points of the orbit of x. The nth iterate of x under f is denoted by x_n. A point $x \in I$ is a *fixed point* of f^n if $f^n(x) = x$ and we say x has *period n* if $f^n(x) = x$ and $f^m(x) \neq x$, $0 < m < n$; in this case the orbit of x is called an *n-cycle* or a *cycle of period n*.

A point x of period n is *locally asymptotically stable* if

$$\lim_{k \to \infty} |f^k(x) - f^k(y)| = 0$$

for near y near x. If f is differentiable a sufficient condition for x to be stable is that the eigenvalue

$$\lambda^{(n)}(x) = \frac{df^n}{dx}(x)$$

has magnitude strictly less than one. If $|\lambda^{(n)}(x)| > 1$, the periodic point x is unstable. Stable periodic orbits will also be referred to as *attractors* or *sinks* and unstable ones as *repellers* or *sources*.

In the following section, we describe the initial bifurcation sequence for the periodic points of f as the parameter is increased from zero and here we describe the three types of bifurcation we encounter. The stability type of a periodic point x can only change at a parameter value $a = a_0$ for which $|\lambda^{(n)}(x)| = 1$ and the generic bifurcations for families of functions with negative Schwarzian derivative, which take place in each of the two cases $\lambda^{(n)}(x) = \pm 1$, are described by Guckenheimer [13]. When $\lambda^{(n)}(x) = +1$ we have a *fold bifurcation* as shown in Fig. 2(a) for the case $n = 1$. Here two periodic orbits, one stable and one unstable, are created as a passes through $a = a_0$. If $\lambda^{(n)}(x) = -1$, we have the *flip bifurcation* pictured in Fig. 2(b) again for $n = 1$. in this case, a stable orbit of period n becomes unstable and gives birth to a stable orbit of period $2n$ at $a = a_0$.

The family f_a also contains examples of the *pitchfork bifurcation*, shown in Fig. 2(c), where a stable orbit of period n becomes unstable by the eigenvalue passing through $+1$ and creates two new stable orbits, each of period n. This bifurcation occurs in situations where the nondegeneracy conditions required for the fold bifurcation and violated, perhaps due to the presence of certain symmetries in the map under consideration. Here we do not state the precise local conditions required for a pitchfork bifurcation to take place since in the only example we meet the bifurcating periodic orbits and their stability can be determined analytically.

2. A CASCADE OF PAIRS OF 2^n-CYCLES

In this section we summarize the salient features of the cubic family

$$f(x) = f_a(x) = ax^3 + (1 - a)x$$

and discuss its bifurcation properties as previously outlined by May. The map has zeros at $x = 0$ and $\pm\sqrt{(a-1)/a}$ and fixed points at $x = 0$ and ± 1. For $a > 1$ there are two critical points at $x = \mp\sqrt{(a-1)/3a}$ with corresponding critical values $\pm\frac{2}{3}\sqrt{(a-1)^3/3a}$. The magnitude of zeros and the critical points increase with a for $a > 0$, and when $a = 4$ the critical values are ± 1 so that f_4 maps the interval I onto itself.

When $a > 0$, the fixed points ± 1 on the boundary of the interval are unstable. To determine the stability of the fixed point at the origin note that the eigenvalue $\lambda^{(1)}(0) = 1 - a$, so the origin is a stable fixed point when $0 < a < 2$. When $a = 2$, $\lambda^{(1)}(0) = -1$ and as a increases

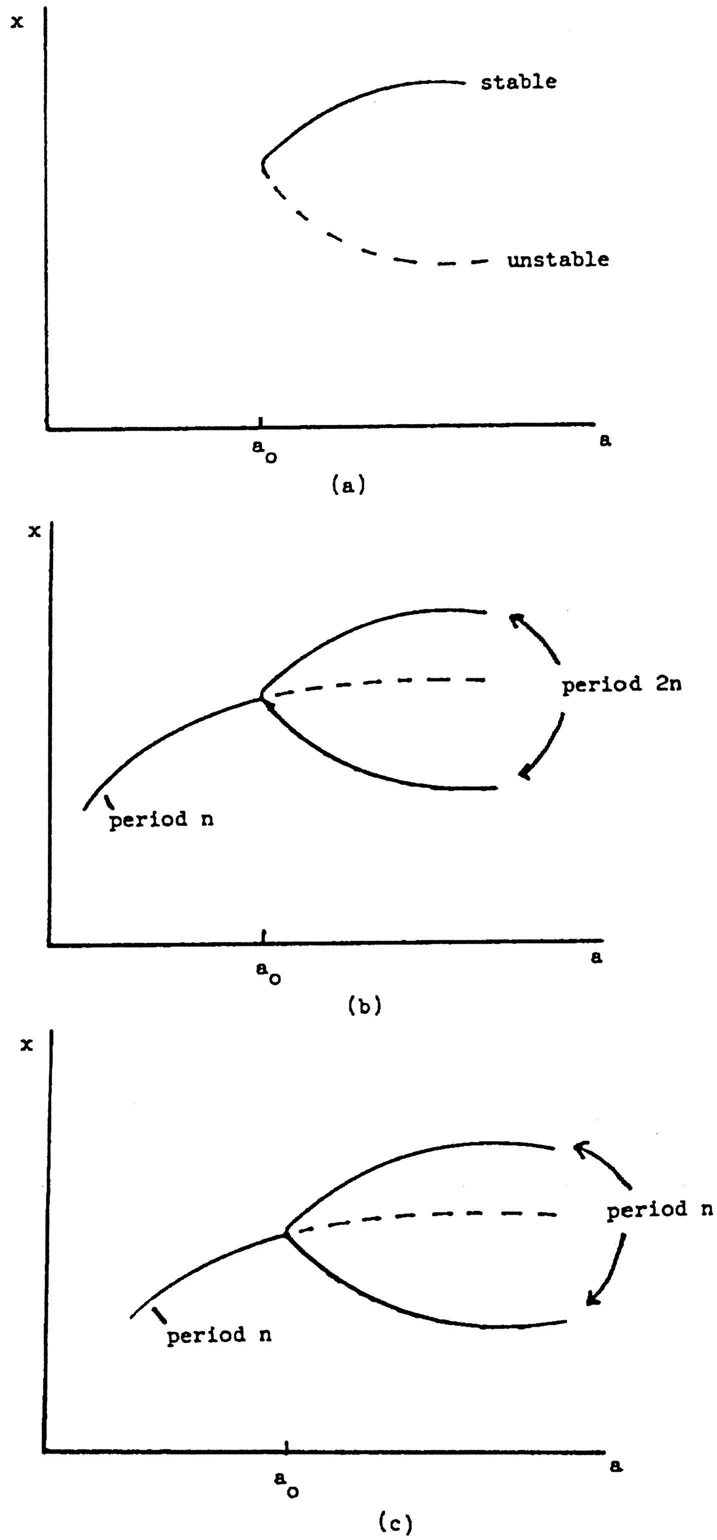

Fig. 2. Fold Bifurcation (a); flip bifurcation (b); pitchfork bifurcation (c). Solid line line = stable and dashed line = unstable.

beyond 2, a flip bifurcation occurs; the origin becomes unstable and a stable orbit of period two is created.

The bifurcating orbit of period two may be found as follows. Since the map f is an odd function, $f(-x) = -f(x)$, for each periodic orbit

$$x_1 \to x_2 \to x_3 \to \cdots \to x_n \to x_1$$

there is a corresponding orbit

$$-x_1 \to x_2 \to \cdots \to -x_n \to -x_1.$$

Now in the flip bifurcation only a single orbit of period two is created so the two orbits above (with $n = 2$) must coincide, and it follows that the orbit has the form

$$\Delta \to -\Delta \to \Delta.$$

Thus the new fixed points of f^2 satisfy $f(\Delta) = -\Delta$; i.e., $\Delta = \sqrt{(a-2)/2}$.

The flip bifurcation from a stable fixed point to a stable orbit of period two is familiar from the many studies of the quadratic map $x \to ax(1-x)$ (or some version of it) but the sequence of period doubling bifurcation in this family does not immediately occur in the cubic family. Instead, the antisymmetry in the map causes the next bifurcation to be a pitchfork. The eigenvalue of f^2 at $\pm\Delta$ is calculated by the chain rule to be $\lambda^{(2)}(\pm\Delta) = (2a-5)^2$. At $a = 2$, the eigenvalue is $+1$ (see Fig. 3). It decreases to zero at $a = 2.5$ and increases back to $+1$ at $a = 3$. As a increases through 3, four new fixed points of f^2 are formed as indicated by Fig. 4; i.e., a pair of 2-cycles are created which are stable for a slightly larger than 3.

These new period two points can also be found explicitly. The fixed points of f^2 satisfy $f^2(x) = x$ and this leads to a polynomial of degree 9 in x from which we can cancel the factor $x(x^2-1)(ax^2+2-a)$ corresponding to the fixed points and the first orbit of period two. This leaves the quartic

$$a^2x^4 + a(1-a)x^2 + 1 = 0$$

or

$$x^2 = \frac{1}{2a}\left[(a-1) \pm \sqrt{(a-3)(a+1)}\right].$$

Let the four solutions be denoted by $x = \pm\alpha, \pm\beta$, then it is easy to verify that the pair of 2-cycles created by the pitchfork are

$$\alpha \to -\beta \to \alpha \quad \text{and} \quad \beta \to -\alpha \to \beta.$$

The eigenvalue of f^2 at each of the points $x = \pm\alpha, \pm\beta$ is found to be $7 \pm 4a - 2a^2$. Thus the period two orbits are stable in the parameter range $3 < a < 1 + \sqrt{5} \approx 3.236\ldots$. At $a = 1 + \sqrt{5}$ the eigenvalue is -1 and as a increases beyond this value a flip bifurcation occurs. The two period two orbits become unstable and a pair of initially stable period four orbits are created. According to numerical simulations carried out by May, as a increases further a sequence of flip bifurcations occurs from each of the stable orbits, producing a hierarchy of pairs of subharmonic cycles of period 4, 8, 16, etc.

3. THE BIFURCATION DIAGRAM

We now turn to the computer-drawn bifurcation diagram shown in Fig. 5. This consists of the last 50 of 250 iterates of each of the two critical points plotted against the parameter

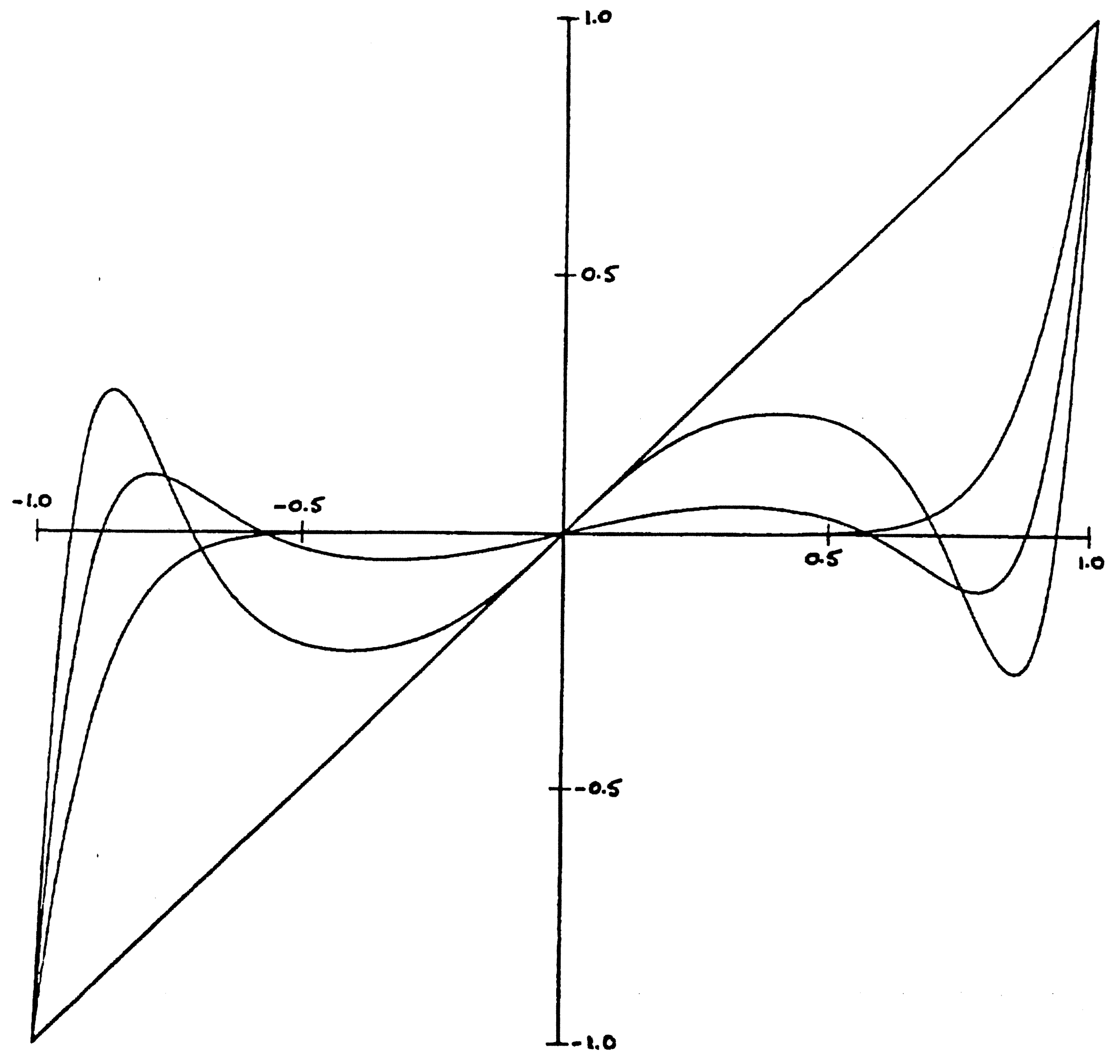

Fig. 3. Graph f_a^2; $a = 1.0, 1.5, 2.0$.

for 500 equally spaced values of the parameter in the range $3 \leq a \leq 4$. The reason for iterating the critical points is that for $a > 1$, f_a has negative Schwarzian derivative and, from the result of Singer [14], it is known that for such a map any stable orbit attracts at least one of the critical points. Thus the orbits plotted in the figure do approach any stable periodic orbits which may exist. Note also that Singer's theorem implies that f_a can have at most two stable periodic orbits for any particular parameter value, and as we saw in Sec. 2, f_a does indeed have two distinct attractors for some a. However, this is not the case for all values of a in the range $3 \leq a \leq 4$, and the bifurcation diagram suggests there is a single stable orbit of period four for $a \approx 3.83$, as we can easily show.

We look for a parameter values for which f has an orbit of period four containing both the critical points; the orbit is then necessarily stable and is also the *unique* attractor for this parameter value and must take the form

$$x_c \to f(x_c) \to -x_c \to f(-x_c) \to x_c,$$

where $\pm x_c$ are the critical values. For such a cycle, we have $f^2(x_c) = -x_c$, which leads to the

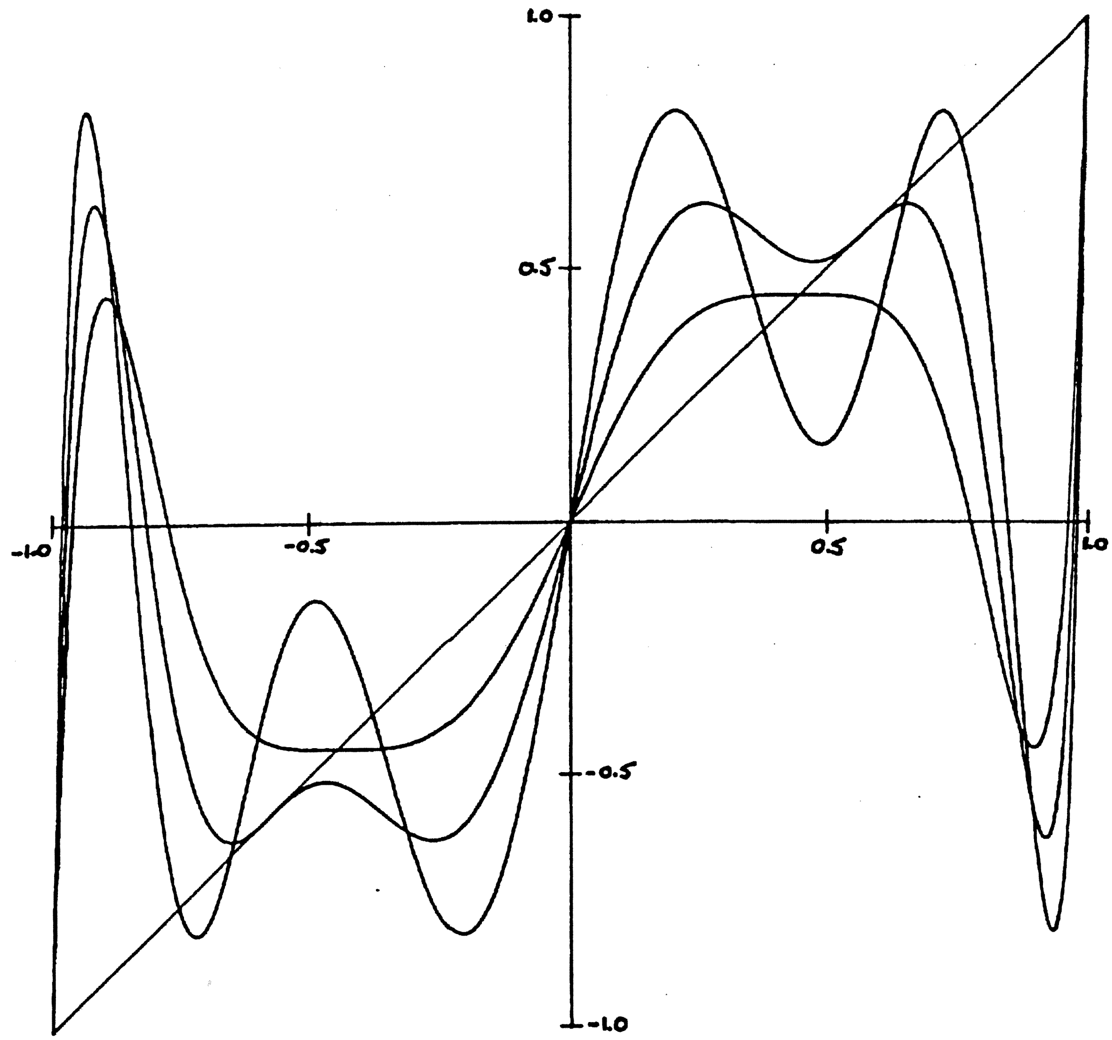

Fig. 4. Graph f_a^2; $a = 2.5, 3, 3.5$.

equation $8a^4 - 32a^3 - 6a^2 + 76a - 127 = 0$ for a. This has a root at $a = 3.8308\ldots$ giving the orbit $0.4963\ldots \to 0.9366\ldots \to -0.4963\ldots \to -0.9366\ldots$, which is the stable 4-cycle visible on the bifurcation diagram. Using the fact that f is an odd function it is easy to see that there is a unique attracting 4-cycle for a near $3.8308\ldots$. We can also find the parameter value at which the orbit is created. Any orbit of the form

$$x \to y \to -x \to -y \to x$$

clearly has $f^2(x) = -x$, i.e.,

$$a[ax^3 + (1-a)x]^3 + (1-a)[ax^3 + (1-a)x] = -x.$$

Let $z = ax^2 + 1 - a$, then we have

$$z^4 + (a-1)z^3 + (1-a)z + 1 = 0.$$

Now if the orbit is created by a fold bifurcation it exists along with an unstable 4-cycle and

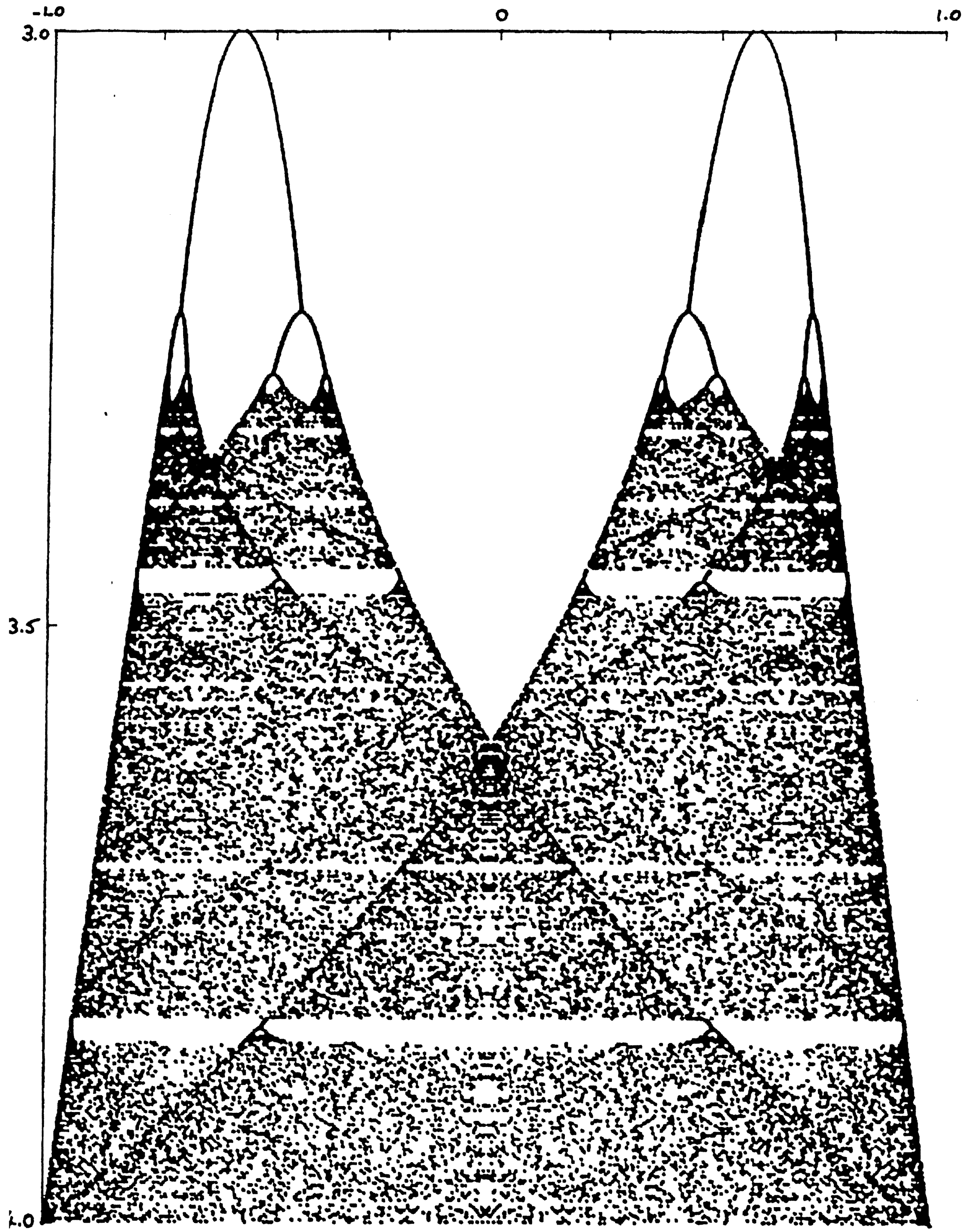

Fig. 5. The bifurcation diagram for $3 \le a \le 4$. A plot of high-order iterates of each of the two critical points.

the two orbits coincide when the bifurcation occurs. At this parameter value we have only four solutions of $f^2(x) = -x$, i.e., only two solutions for z, opposed to eight solutions of $f^2(x) = -x$ and four solutions for z at slightly larger parameters. Thus at the point of bifurcation where the 4-cycle is created the polynomial for z takes the form $(z^2 + Az - 1)^2 = 0$, and this is true iff $A = (a-1)/2$ and $a^2 - 2a - 7 = 0$; i.e., $a = 1 + 2\sqrt{2}$. Thus f_a has a single stable 4-cycle for $1 + 2\sqrt{2} < a < a_+$, where $a_+ > 3.8308\ldots$. Also from the diagram it appears there may exist a single stable 6-cycle at $a \approx 3.4$, but this has not been pursued and it is hard to see any general scheme for deciding which stable orbits occur in pairs and which occur singly.

The bifurcation diagram suggests that there is an accumulation point of the values of a which correspond to the bifurcations in the initial period-doubling sequence. It also seems likely that the rate of convergence of these parameters values to their limit will involve the universal constants found by Feigenbaum [15] in families of maps with a single critical point, but no attempt has been made to verify this.

It is important to remember that Fig. 5 contains two orbits for each parameter value; we note also that points in the two major branches which emerge in the first sequence of flip bifurcations and appear to be two disjoint attractors, one in $[-1, 0]$ and the other in $[0, 1]$, for $a \leq 3.6$ approximately, are, in fact, mapped from one branch to the other. To see this consider the subintervals $J_1 = [-\sqrt{(a-1)/a}, 0]$ and $J_2 = [0, \sqrt{(a-1)/a}]$ whose boundaries are the zeros of f_a. For $x \in (-1, 0)\backslash J_1, f(x) > x$, and for $x \in (0, 1)\backslash J_2, f(x) < x$ so that all points eventually map into $J_1 \cup J_2$. Also, provided the magnitude of the critical values of f_a is less than or equal to that of the zeros, f maps J_1 into J_2 and vice versa. Thus in this case $J_1 \cup J_2$ forms a traping region which all points in $(-1, 1)$ enter and never leave. This behaviour persists up to the parameter value where the critical values and the zeros of f_a coincide; i.e., the value of a for which the second iterate of each critical point is precisely the unstable fixed point at the origin. This value of $a = a^* = 1 + \frac{1}{2}\sqrt{27} = 3.5980\ldots$ is the point at which the two major branches in the bifurcation diagram coalesce and marks an important change in the dynamics of the family f_a (see Fig. 5). For $a < a^*$, f_a has no periodic points of odd period since the nonwandering set of $f|_{(-1,1)}$ is wholly contained in $J_1 \cup J_2$, and f_a maps points from each of these subintervals into the other. For $a > a^*$, we show below that the origin is a snapback repeller for f_a and hence the recurrence $x_{n+1} = f_a(x_n)$ has chaotic solutions.

4. A SNAPBACK REPELLER

We now come to our main result. We show that for $a > a^*$ the cubic family f_a admits solutions which are chaotic in the Li–Yorke sense [6]: there are (i) periodic solutions of all orders, N, $N+1\ldots$ beyond a given order, and (ii) there are an uncountable number of nonperiodic orbits that are not even asymptotically periodic which, for $x \neq y$, satisfy the "near-far" conditions $0 = \liminf |f^n(x) - f^n(y)| < \limsup |f^n(x) \times f^n(y)|$. The well-known Li–Yorke theorem says that if a point x satisfies $f^3(x) \leq x < f(x) < f^2(x)$, then f is chaotic, in fact with $N = 1$, i.e., all periods exist. The corollary "period 3 implies chaos" follows. Li *et al.* [16] have recently improved this result: if there exists an x satisfying $f^n(x) < x < f(x)$ for some odd $n > 1$, then f has a point of odd periods k, $1 < k \leq n$, where k divides n; the existence of such a point of odd period $k > 1$ implies conditions (i) and (ii), i.e., Li–Yorke chaos. Rather than search for odd periods directly, we turn to a sufficiency condition formulated by Marotto [11] which appears to be remarkably useful. Chaos has been verified in a variety of first and second-order difference equation models arising in population dynamics, using this condition. The condition is an n-dimensional one, and it is most convenient to state it as such, even though our map is only one-dimensional.

Let $f: \mathbf{R}^n \to \mathbf{R}^n$ and assume f is differentiable in some ball $B_r(z)$ of radius r about z. The point z is called an *expanding fixed point* of f if $f(z) = z$ and all eigenvalues of the derivative

$Df(x)$ exceed 1 in magnitude for all $x \in B_r(z)$. The repelling fixed point z is called a *snapback repeller* if there is a point $x_0 \in B_r(z)$ with $x_0 \neq z$, $f^M(x_0) = z$, and $\|Df^M(x_0)\| \neq 0$ for some positive integer $M > 1$.

Marotto's theorem states that snapback repellers imply chaos in $\boldsymbol{R}^n$. We sketch the following proof: Since $\|Df^M(x_0\| \neq 0, f^M$ is invertible in a ball $B_\epsilon(z)$ centered at z and contained in $B_r(z)$. $B_\epsilon(z)$ maps to a neighborhood $Q \subset B_r(z)$ of x_0 by a map whose restriction to $B_\epsilon(z)$ is f^{-M}. Then since z is repelling, all sufficiently high order iterates of f^{-1}, say $f^{-(M+N)}$ for $N > \bar{N}$, map Q back into $B_\epsilon(z)$. Thus $f^{-(M+N)}$ has a fixed point in $B_\epsilon(z)$ by the Brouwer fixed point theorem. This point is a fixed point of $f^{(M+N)}$ as well.

We proceed to determine conditions which make the fixed point 0 a snapback repeller for the cubic map. When $a > 2, f'(0) = 1 - a < 1$, so 0 is repelling for all $a > 2$. We wish to find an interval $B_r(0) = I = [-r, r]$ with $f'(x) < -1$ for all $x \in I$, and such that the other snapback repeller conditions are met. We calculate $f'(x) = 3ax^2 + (1 - a) < -1$ iff $|x| < ((a - 2)/3a)^{1/2}$. Denote $g(a) = ((a - 2)/3a)^{1/2}$.

In searching for preimages of 0 which lie near 0 it is reasonable to find conditions which guarantee that the first few preimage sets of 0 are nonempty. Referring to Fig. 6, the only

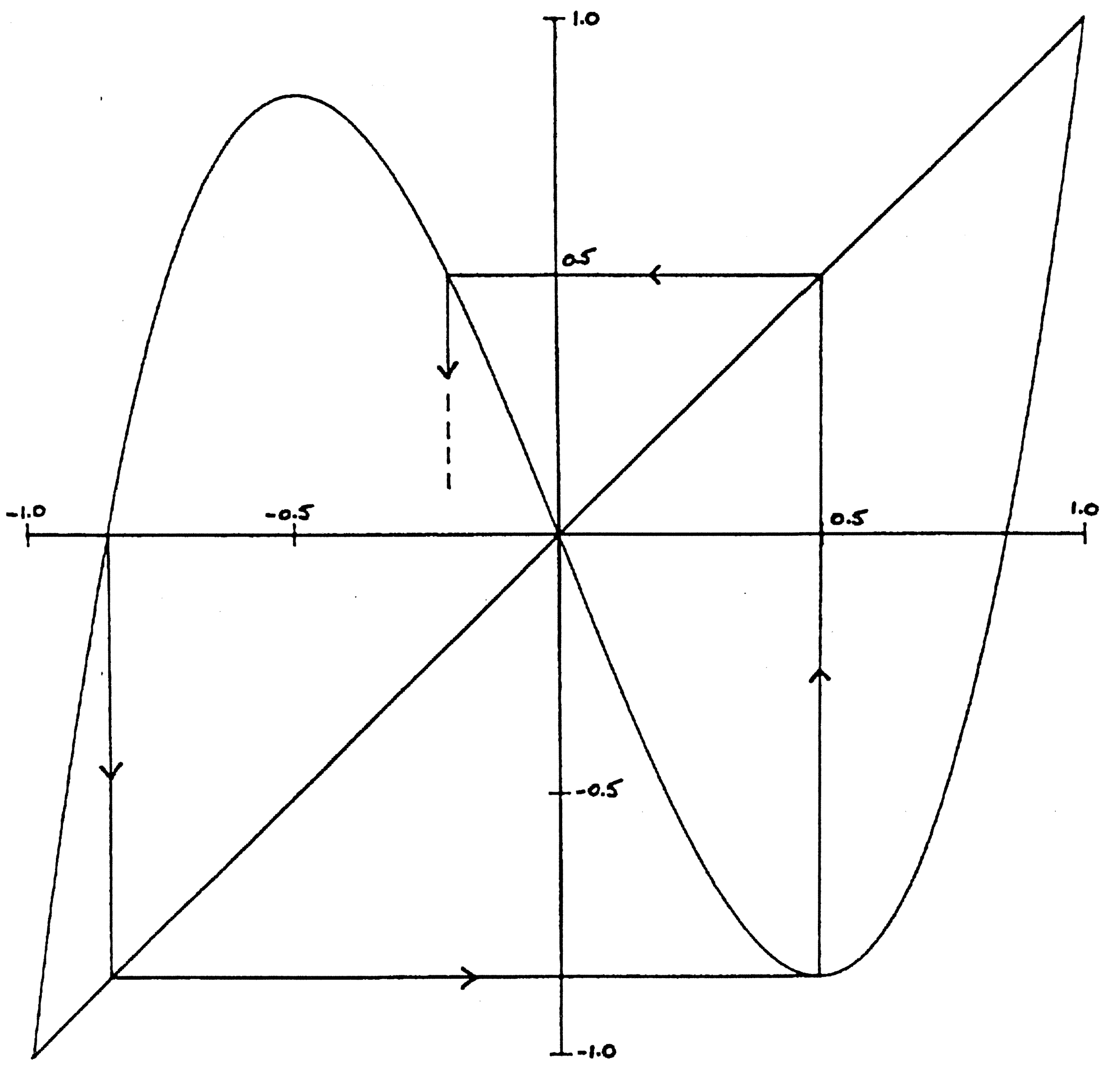

Fig. 6. Graph f_a; $a = a^* = 3.5980\ldots$. The directed line indicates a trajectory of inverse images of 0.

negative preimage of 0 is the zero $-((a-1)/a)^{1/2}$. Evidently there is some critical value of the parameter a for which this negative zero in turn has some positive preimage, i.e., $f^{-2}(0)$ is nonempty and has positive members; geometrically, we are saying the curve must be steep enough. In fact the critical value of the parameter is precisely the value $a^* = 1 + \sqrt{27}/2 = 3.5980\ldots$ found in the prevous section for which the set of second preimages of 0 contains the critical points.

The bound on $|x|$ for which $f'(x) < -1$ becomes in the critical case

$$r = g(a^*) = ((a^* - 2)/3a^*)^{1/2} = 0.38477\ldots$$

Since g is increasing it follows that

$$f'(x) < -1 \quad \text{for all} \quad |x| < r \quad \text{when} \quad a > a^*.$$

We denote $I = [-r, r]$, and calculate certain preimages of 0 when $a = a^*$ (see Fig. 6); the selected preimages will make a counterclockwise spiral into a neighborhood of the fixed point. The multivalued inverse iterates solve the recurrence

$$x_{k-1}^3 + \left(\frac{1-a}{a}\right)x_{k-1} - \frac{1}{a}x_k = 0, \qquad k = 0, -1\ldots$$

Consideŗ the case $a = a^*$.

Take $x_0 = 0$ and choose the negative preimage of 0 which from above is $x_{-1} = -((a^* - 1)/a^*)^{1/2} = -0.84974\ldots$. For the critical point x_{-2}, we choose the positive root of $x^3_{-2} + ((1 - a^*)/a^*)x_{-2} + (1/a^*)((a^* - 1)/a^*)^{1/2} = 0$ to obtain $x_{-2} = 0.49185\ldots$. Similarly we choose the biggest negative root $x_{-3} = -0.19596\ldots$ Note that $x_{-3} \in I$.

Observe that the transversality condition $|Df^M(x_{-M})| \neq 0$ is violated for $M \geq 1$ precisely at the critical parameter value a^* because the preimage x_{-2} is a critical point. We will show that for $a > a^*$, 0 is a snapback repeller in I, and in fact we may take $M = 3$ for all such a. For $a > a^*$, we choose the iterates $x_{-k}(a)$ in a similar manner as when $a = a^*$ so that the preimages spiral toward 0 counterclockwise, and always the preimage closest to 0 is chosen which has opposite sign to the previous one.

We show that $x_{-3} = x_{-3}(a) \in I$ for all $a > a^*$. For this it is enough to show that $x_{-2} = x_{-2}(a)$ decreases with a when $a > a^*$. In this parameter range, none of the iterates x_{-1}, $x_{-2}, x_{-3}\ldots$ can be critical points, and the snapback repeller conditions would be verified with $M = 3$ for all $a > a^*$. The derivative of x_{-2} is

$$\frac{\mathrm{d}}{\mathrm{d}a}(x_{-2}(a)) = \frac{\dfrac{x_{-2}(a)}{a^2} - \dfrac{1}{2}\left(\dfrac{a^3}{a-1}\right)^{1/2}\left(\dfrac{3-2a}{a^4}\right)}{3x^2_{-2}(a) + \left(\dfrac{1-a}{a}\right)}.$$

The numerator of this expression is positive if $a > a^*$. The denominator is negative if $x_{-2}(a) < ((a-1)/3a)^{1/2}$. This latter condition is satisfied because we have defined $x_{-2}(x)$ to be the least positive root when $k = 3$, and $((a-1)/3a)^{1/2}$ is the critical point corresponding to the negative minimum value of f_a. Therefore, $x_{-2}(a)$ decreases with a when $a > a^*$. We infer that $x_{-3}(a^*) < x_{-3}(a) < 0$, and hence $|x_{-3}(a)| < g(a^*) = r$. Then $x_{-3}(a) \in I$, and we have verified the snapback repeller conditions for the cubic map with $M = 3$, $r = 0.38471\ldots$ for all $a > a^* = 3.5980\ldots$ We may state the following theorem.

Theorem. The cubic map has the fixed point 0 at a snapback repeller, and is thus chaotic, for all $a > a^*$.

We remark that this is the best possible result of this type since f_a has no snapback repeller if $a < a^*$. This follows from the fact that the existence of a snapback repeller implies the existence of periodic points of all periods larger than some integer N. In particular, f must have an orbit of odd period > 1, but as noted above this is impossible when $a \leq a^*$. The geometry of this situation is typified by the graph of $f^2_{a^*}$ in Fig. 7 and the graph of $f^3_{a^*}$ in Fig. 8.

There is a further connection between the parameter value a^* above which the snapback repeller exists, and the existence of odd periodic orbits. Given a parameter value $a = a^* + \epsilon$, $\epsilon > 0$, by the theorem, f_a has a snapback repeller at the origin and hence an orbit of some odd period M, say. By Sarkovskii's Theorem (see Stefan [17]), f_a then has points of all even periods together with points of all odd periods greater than M. For each odd integer $k > M$, let a_k denote the least parameter value for which f_a has an orbit of period k. Now if we make the assumption that the a_k are the parameter values at which orbits of period k are created by fold bifurcations then no two of the a_k may coincide, as it follows from Singer's Theorem and the fact that f_a is an odd function that no two stable orbits of different periods can exist for the same parameter value. Thus for $k > N$ we have $a^* < a_k < a_M$ and a^* is an

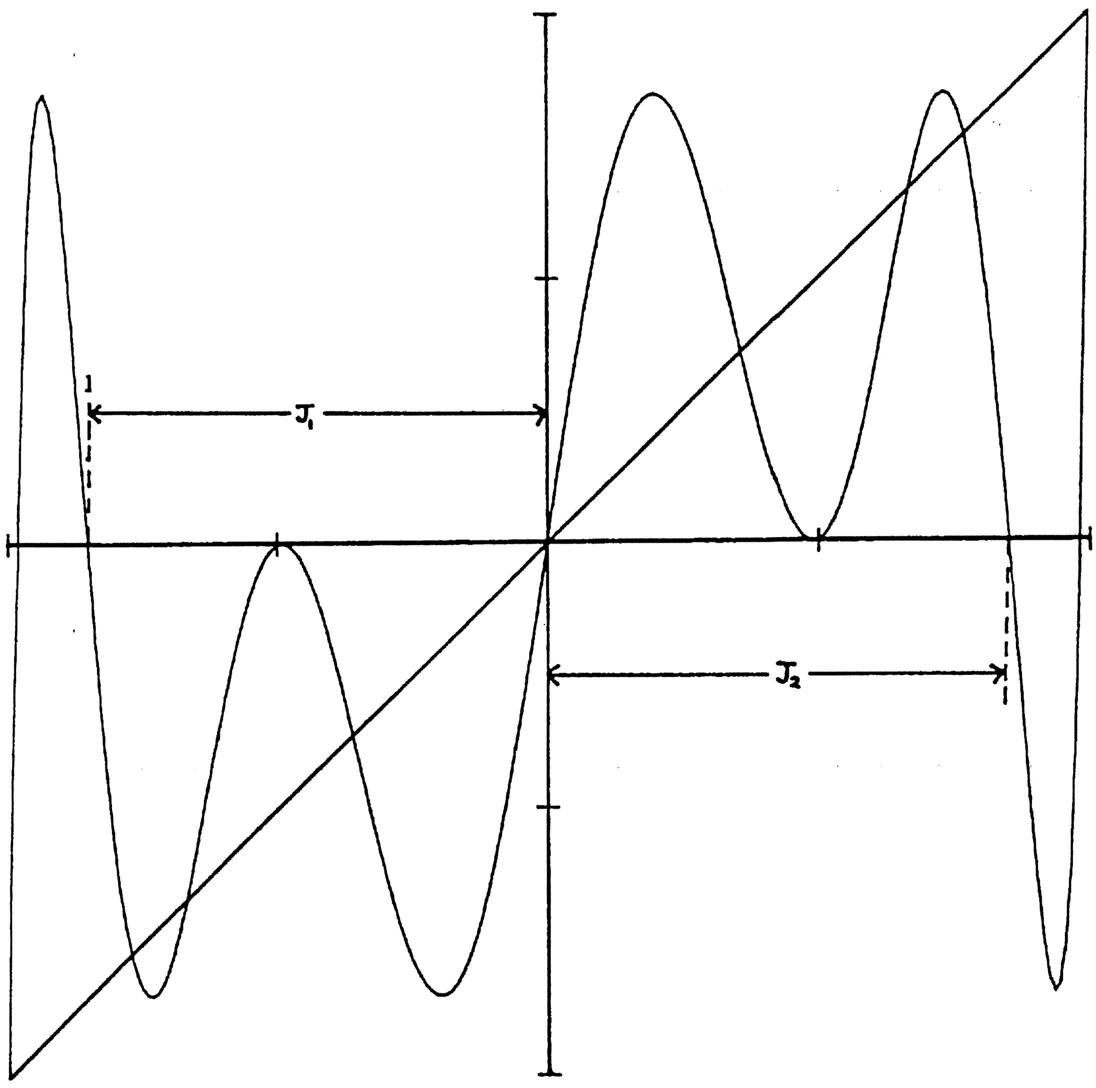

Fig. 7. Graph $f^2_{a^*}$: $a = a^* = 3.5980\ldots$

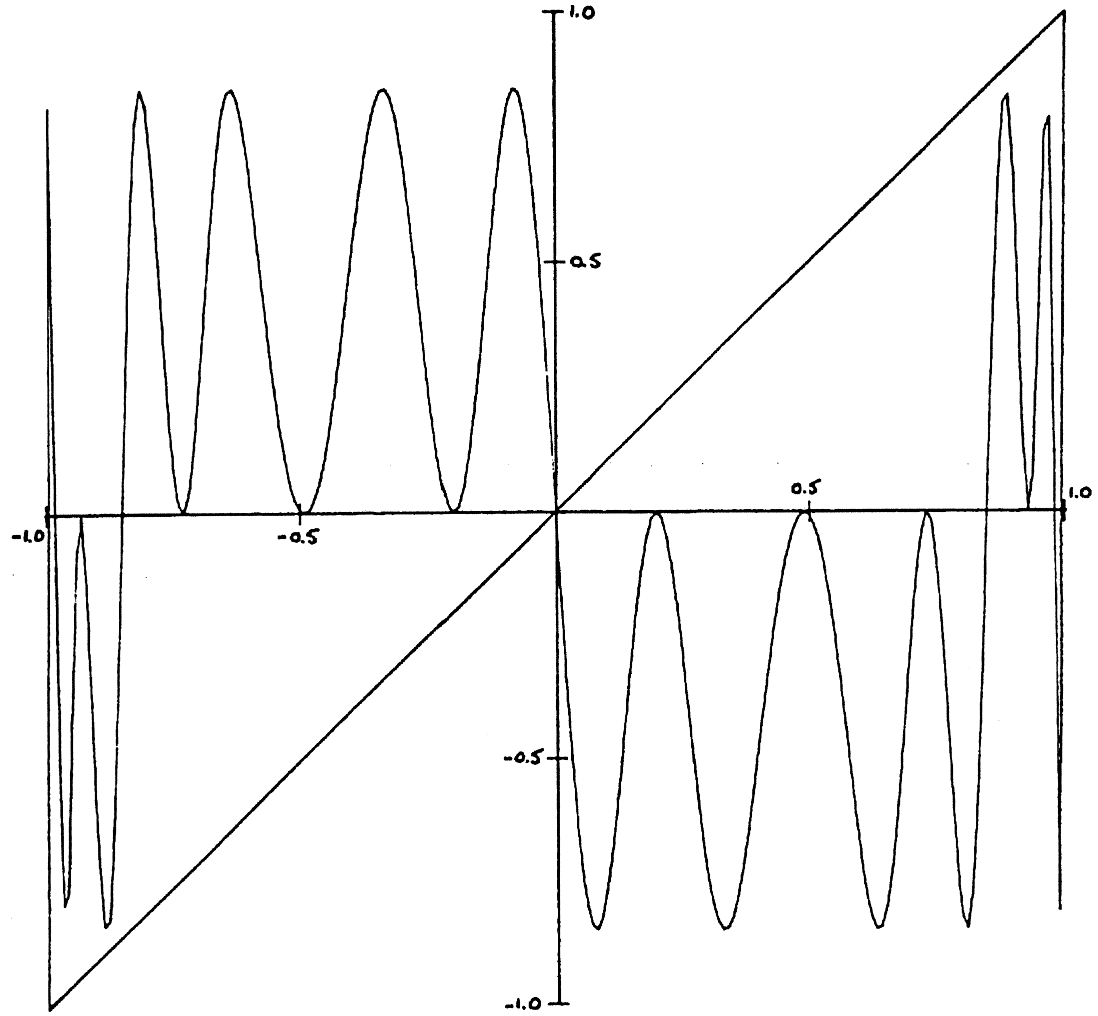

Fig. 8. Graph $f^3_{a^*}$; $a = a^* = 3.5980\ldots$. There are no points of odd period > 1.

accumulation point for the sequence of a_k which is approached from above. In this sense a^* is the parameter value where the "first" odd periodic orbit appears.

Now it is clear from the bifurcation diagram that the map has chaotic solutions for values of a less than a^*, and a similar analysis to that above, but this time searching for snapback repellers in successively higher iterates of the map, would possibly lead to a sequence of parameters a_n^* corresponding to the appearance of a snapback repeller for f^n. The sequence will then converge to the parameter value at which chaos first appears. [This is the approach taken by Marotto [11] in his study of the quadratic family $x \to ax(1-x)$.] However, the calculations involved in carrying out this program are sufficiently complicated that the analysis soon becomes intractable.

5. THE DYNAMICS OF f_{a^*} AND f_4

In this section, we study the map f_a for two special parameter values where an interval is mapped onto itself. The first case is $a = a^*$, the parameter value at which the snapback repeller appears, and the second is $a = 4$, the extreme value beyond which points are mapped out of the interval $I = [-1, 1]$.

When $a = a^*$, $f = f_{a^*}$ maps the subinterval J_1 onto J_2 and vice versa, so that f^2 folds J_1

(or J_2) onto itself 4 times, as indicated in Fig. 7. The number of cycles of $f^2|_{J_1}$ can be counted using the technique of symbolic dynamics as explained by Guckenheimer [13] and Kaplan and Yorke [18]. The three critical points of f^2 in J_1 partition J_1 into four subintervals which we label in order α, β, γ, δ. Each of these intervals is mapped onto J_1 by f^2 so the action of f^2 on J_1 may be represented by the transition matrix

$$M = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

where

$$M_{ij} = \begin{cases} 1 & \text{if } f^2 \text{ maps a point in } i \text{ into } j \\ 0 & \text{otherwise.} \end{cases}$$

Here we have $M_{ij} = 1$ for each $i, j = \alpha, \beta, \gamma, \delta$.

The number of fixed points of the kth iterate of $f^2|_{J_1}$ is given by the trace formula

$$\text{Trace}(M^k) = 4^k,$$

and the number of distinct k-cycles is obtained by subtracting from $\text{Tr}(M^k)$ the number of points whose period divides k, and then dividing by k (each point in a cycle is counted separately in the trace formula). We may then successively compute the number of k-cycles, and the first few are tabulated below.

k	1	2	3	4	5	6
No. of cycles of $f^2\|_{J_1}$ of period k	4	6	20	60	204	670

Each k-cycle of $f^2|_{J_1}$ except the fixed point of f at $x = 0$, corresponds to a $2k$-cycle of $f|_{J_1 \cup J_2}$. Also f has no periodic orbits of odd period > 1 so the number of k-cycles of $f|_{[-1,1]}$ are easily seen to be as follows

k	1	2	3	4	5	6
No. of cycles of $f\|_{[-1,1]}$ of period k	3	3	0	6	0	20

The same analysis can be applied to the case $a = 4$ where $f = f_4$ folds the interval $I = [-1, 1]$ onto itself three times. Here, partitioning the I by the two critical points leads to a 3×3 transition matrix M, all of whose entries are 1. The number of k-cycles can then be found from the formula

$$\text{No. of fixed points of } f^k = \text{Tr}(M^k) = 3^k,$$

and a table such as that above is easily produced. However, when $a = 4$ we have a much more detailed description of the dynamics of f.

We show the cubic map f_4 is topologically equivalent to a piecewise linear map with constant slope 3 in magnitude. Since the slope is greater than 1, the map is completely chaotic; there can be no stable cycles. It turns out that $f_4(x) = 4x^3 - 3x$ has the same invariant measure as the much-studied quadratic map $g(x) = 1 - 2x^2$ defined on $[-1, 1]$, found by Ulam and Von Neumann [19]. The reason the two maps have a common measure is interesting. Both maps are Chebyshev polynomials, a class of orthogonal polynomials T_n, defined on $[-1, 1]$ in this case, each of which preserves the same measure. The quadratic map g is T_2 and the cubic map f_4 is T_3.

The ergodic properties of the Chebyshev polynomials are described by Alder and Rivlin [12] and we recall some of their results here. The Chebyshev polynomial of degree n is

$$T_n(x) = \cos(n\theta), \quad \text{where} \quad x = \cos(\theta), \qquad 0 \le \theta \le \pi.$$

Denote $X = [-1, 1]$ and $\mathscr{B}$ = the family of Borel sets of X equipped with Lebesque measure. Define the measure μ as

$$\mu(B) = \frac{2}{\pi} \int_B \frac{dx}{\sqrt{1 - x^2}}, \quad B \in \mathscr{B}.$$

Then μ is absolutely continuous with respect to Lebesgue measure. With $T_0(x) = 1/\sqrt{2}$, the T_n, $n = 0.1\ldots$ form a complete orthonormal set in X. Each T_n maps X onto X and is measurable almost everywhere. For our particular application, we have

$$T_3(x) = T_3[\cos(\theta)] = \cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta) = 4x^3 - 3x = f_4(x).$$

Adler and Rivlin proved that each T_n, $n > 1$, preserves the measure μ, i.e.,

$$\mu(T_n^{-1}(B)) = \mu(B), B \in \mathscr{B}, \qquad n = 1, 2\ldots$$

Thus any finite distribution of points on X will have the same density $2/\pi\sqrt{1 - x^2}$ after any number of iterations of the map T_n [20]. Further each T_n is *strongly mixing* in the sense that

$$\lim_{k \to \infty} [\mu(T_n^{-k} A \cap B)] = \frac{\mu(A)\mu(B)}{\mu(X)}$$

From the measure-preserving condition this is equivalent to

$$\mu(T_n^{-k} A / B) - \mu(T_n^{-k} A) \to 0,$$

as $k \to \infty$, so the mixing property refers to the "wearing off" of initial conditions [21]; initial points under T_n are stirred over the interval.

A corollary of the mixing property is that the T_n, $n > 1$, are ergodic transformations, i.e., $T_n^{-1}B = B$ implies $\mu(B) = 0$ or $\mu(B) = X$. By the fundamental theorem of ergodic theory, this implies that the trajectory of each point x of X enters B with limiting frequency $\mu(B)$.

The key idea behind Rivlin's proof that the T_n preserves the measure is the fact that the transformations are conjugate to piecewise linear maps. Let R be the invertible map $R(x) = \arccos(x)$ from X to $[0, \pi]$, and define $S_n = RT_nR^{-1}$. Partition $[0, \pi]$ by $k\pi/n \le x \le (k + 1)\pi/n$, $k = 0, 1 \ldots n - 1$. Then $S_n(x) = nx - k\pi$, k even and $S_n(x) = -nx + (k + 1)\pi$, k odd. S_n preserves Lebesgue measure. As simple calculation then leads to the result that the T_n preserve μ.

6. CONCLUDING REMARKS

(1) The known examples [9] for which maps of the interval have invariant measure have the property that a critical point is mapped to an unstable fixed point. It is tempting to conjecture that this property will always imply the existence of an invariant measure. Note the map f_{a^*} of this paper satisfies the condition. For results along this line for unimodal functions with negative Schwarzian derivative, see Misiurewicz [22] and Jakobson [23].

(2) Adler and Rivlin [12] point out that the strong mixing of T_n can be deduced from the stong mixing of the discontinuous n-adic transformation Q_n: $x \to nx(\text{mod } 1)$ defined on $[0, 1]$ (see Billingsley [21]).

(3) May [2] outlined a reasonable set of problems which would extend the analysis of f_a beyond his investigation of the map as outlined in Sec. 2. Although our paper does provide a partial description of the chaotic region, with emphasis on the parameter value a^* at which odd periods first occur, there remains much to be done in describing the bifurcations of the cubic map. Specifically a numerical analysis, verifying May's conjecture that the Feigenbaum ratios remain the same for maps with more than one critical point, would be a useful achievement. May also showed how the cascade of pairs of 2^n-cycles is generic with respect to the family of smooth maps f with two critical points which, in addition, are antisymmetric, i.e., $f(-x) = -f(x)$. We suspect that the major results of this paper carry over easily to the general antisymmetric case as well.

(4) In terms of applications, May [2] explains how chaotic oscillations are unlikely to occur in plausible one-dimensional genetic models; however, we regard this biologically-motivated example as interesting in its own right as a prototype for the dynamics of maps with more than one critical point.

(5) Since the cubic map is chaotic for $a > a^*$ we may apply Marotto's perturbation theorem [24, 25] to conclude the related planar map $(x, y) \to (ax^3 + (1-a)x + by, x)$ has transverse homoclinic points. This implies chaotic oscillations in the discrete version $x_{n+1} = ax_n^3 + (1-a)x_n + by_n$, $y_{n+1} = x_n$ of the unforced Duffing's equation $\ddot{x} - ax^3 - (1-a)x - b\dot{x} = 0$.

Acknowledgements—We are grateful to Steve Shaw for doing the computer-drawn graphs. The first author thanks Phil Holmes and Simon Levin for their hospitality at the Center for Applied Mathematics at Cornell while this paper was written.

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