EXAMPLE 6.8.6:

Show that the system \( \dot{x} = xe^{-x}, \dot{y} = 1 + x + y^2 \) has no closed orbits.

Solution: This system has no fixed points: if \( \dot{x} = 0 \), then \( x = 0 \) and so \( \dot{y} = 1 + y^2 \neq 0 \). By Theorem 6.8.2, closed orbits cannot exist. 

EXERCISES FOR CHAPTER 6

6.1 Phase Portraits

For each of the following systems, find the fixed points. Then sketch the nullclines, the vector field, and a plausible phase portrait.

6.1.1 \( \dot{x} = x - y, \dot{y} = 1 - e^x \)  
6.1.2 \( \dot{x} = x - x^3, \dot{y} = -y \)

6.1.3 \( \dot{x} = x(x - y), \dot{y} = y(2x - y) \)  
6.1.4 \( \dot{x} = y, \dot{y} = x(1 + y) - 1 \)

6.1.5 \( \dot{x} = x(2 - x - y), \dot{y} = x - y \)  
6.1.6 \( \dot{x} = x^2 - y, \dot{y} = x - y \)

6.1.7 (Nullcline vs. stable manifold) There’s a confusing aspect of Example 6.1.1. The nullcline \( \dot{x} = 0 \) in Figure 6.1.3 has a similar shape and location as the stable manifold of the saddle, shown in Figure 6.1.4. But they’re not the same curve! To clarify the relation between the two curves, sketch both of them on the same phase portrait.

(Computer work) Plot computer-generated phase portraits of the following systems. As always, you may write your own computer programs or use any ready-made software, e.g., MacMath (Hubbard and West 1992).

6.1.8 (van der Pol oscillator) \( \dot{x} = y, \dot{y} = -x + y(1 - x^2) \)

6.1.9 (Dipole fixed point) \( \dot{x} = 2xy, \dot{y} = y^2 - x^2 \)

6.1.10 (Two-eyed monster) \( \dot{x} = y + y^2, \dot{y} = -\frac{1}{2}x + \frac{1}{2}y - xy + \frac{5}{3}y^2 \) (from Borrelli and Coleman 1987, p. 385.)

6.1.11 (Parrot) \( \dot{x} = y + y^2, \dot{y} = -x + \frac{1}{2}y - xy + \frac{5}{6}y^2 \) (from Borrelli and Coleman 1987, p. 384.)

6.1.12 (Saddle connections) A certain system is known to have exactly two fixed points, both of which are saddles. Sketch phase portraits in which

a) there is a single trajectory that connects the saddles;

b) there is no trajectory that connects the saddles.

6.1.13 Draw a phase portrait that has exactly three closed orbits and one fixed point.

6.1.14 (Series approximation for the stable manifold of a saddle point) Recall the system \( \dot{x} = x + e^{-x}, \dot{y} = -y \) from Example 6.1.1. We showed that this system
has one fixed point, a saddle at $(-1,0)$. Its unstable manifold is the $x$-axis, but its stable manifold is a curve that is harder to find. The goal of this exercise is to approximate this unknown curve.

a) Let $(x, y)$ be a point on the stable manifold, and assume that $(x, y)$ is close to $(-1, 0)$. Introduce a new variable $u = x + 1$, and write the stable manifold as $y = a_1u + a_2u^2 + O(u^3)$. To determine the coefficients, derive two expressions for $dy/du$ and equate them.

b) Check that your analytical result produces a curve with the same shape as the stable manifold shown in Figure 6.1.4.

### 6.2 Existence, Uniqueness, and Topological Consequences

6.2.1 We claimed that different trajectories can never intersect. But in many phase portraits, different trajectories appear to intersect at a fixed point. Is there a contradiction here?

6.2.2 Consider the system $\dot{x} = y$, $\dot{y} = -x + (1 - x^2 - y^2)y$.

a) Let $D$ be the open disk $x^2 + y^2 < 4$. Verify that the system satisfies the hypotheses of the existence and uniqueness theorem throughout the domain $D$.

b) By substitution, show that $x(t) = \sin t$, $y(t) = \cos t$ is an exact solution of the system.

c) Now consider a different solution, in this case starting from the initial condition $x(0) = \frac{1}{2}$, $y(0) = 0$. Without doing any calculations, explain why this solution must satisfy $x(t)^2 + y(t)^2 < 1$ for all $t < \infty$.

### 6.3 Fixed Points and Linearization

For each of the following systems, find the fixed points, classify them, sketch the neighboring trajectories, and try to fill in the rest of the phase portrait.

6.3.1 $\dot{x} = x - y$, $\dot{y} = x^2 - 4$

6.3.3 $\dot{x} = 1 + y - e^{-x}$, $\dot{y} = x^3 - y$

6.3.5 $\dot{x} = \sin y$, $\dot{y} = \cos x$

6.3.2 $\dot{x} = \sin y$, $\dot{y} = x - x^3$

6.3.4 $\dot{x} = y + x - x^3$, $\dot{y} = -y$

6.3.6 $\dot{x} = xy - 1$, $\dot{y} = x - y^3$

6.3.7 For each of the nonlinear systems above, plot a computer-generated phase portrait and compare to your approximate sketch.

6.3.8 (Gravitational equilibrium) A particle moves along a line joining two stationary masses, $m_1$ and $m_2$, which are separated by a fixed distance $a$. Let $x$ denote the distance of the particle from $m_1$.

a) Show that $\ddot{x} = \frac{Gm_2}{(x-a)^2} - \frac{Gm_1}{x^2}$, where $G$ is the gravitational constant.

b) Find the particle's equilibrium position. Is it stable or unstable?
6.3.9 Consider the system \( \dot{x} = y^3 - 4x, \quad \dot{y} = y^3 - y - 3x \).

a) Find all the fixed points and classify them.
b) Show that the line \( x = y \) is invariant, i.e., any trajectory that starts on it stays on it.
c) Show that \( |x(t) - y(t)| \to 0 \) as \( t \to \infty \) for all other trajectories. (Hint: Form a differential equation for \( x - y \).)
d) Sketch the phase portrait.
e) If you have access to a computer, plot an accurate phase portrait on the square domain \(-20 \leq x, y \leq 20\). (To avoid numerical instability, you’ll need to use a fairly small step size, because of the strong cubic nonlinearity.) Notice the trajectories seem to approach a certain curve as \( t \to -\infty \); can you explain this behavior intuitively, and perhaps find an approximate equation for this curve?

6.3.10 (Dealing with a fixed point for which linearization is inconclusive) The goal of this exercise is to sketch the phase portrait for \( \dot{x} = xy, \quad \dot{y} = x^2 - y \).

a) Show that the linearization predicts that the origin is a non-isolated fixed point.
b) Show that the origin is in fact an isolated fixed point.
c) Is the origin repelling, attracting, a saddle, or what? Sketch the vector field along the nullclines and at other points in the phase plane. Use this information to sketch the phase portrait.
d) Plot a computer-generated phase portrait to check your answer to (c).

(Note: This problem can also be solved by a method called center manifold theory, as explained in Wiggins (1990) and Guckenheimer and Holmes (1983).)

6.3.11 (Nonlinear terms can change a star into a spiral) Here’s another example that shows that borderline fixed points are sensitive to nonlinear terms. Consider the system in polar coordinates given by \( \dot{r} = -r, \quad \dot{\theta} = 1/\ln r \).

a) Find \( r(t) \) and \( \theta(t) \) explicitly, given an initial condition \((r_0, \theta_0)\).
b) Show that \( r(t) \to 0 \) and \( |\theta(t)| \to \infty \) as \( t \to \infty \). Therefore the origin is a stable spiral for the nonlinear system.
c) Write the system in \( x, y \) coordinates.
d) Show that the linearized system about the origin is \( \dot{x} = -x, \quad \dot{y} = -y \). Thus the origin is a stable star for the linearized system.

6.3.12 (Polar coordinates) Using the identity \( \theta = \tan^{-1}(y/x) \), show that \( \dot{\theta} = (xy - y\dot{x})/r^2 \).

6.3.13 (Another linear center that’s actually a nonlinear spiral) Consider the system \( \dot{x} = -y - x^3, \quad \dot{y} = x \). Show that the origin is a spiral, although the linearization predicts a center.

6.3.14 Classify the fixed point at the origin for the system \( \dot{x} = -y + ax^3, \quad \dot{y} = x + ay^3 \), for all real values of the parameter \( a \).
6.3.15 Consider the system \( \dot{r} = r(1 - r^2), \; \dot{\theta} = 1 - \cos \theta \), where \( r, \theta \) represent polar coordinates. Sketch the phase portrait and thereby show that the fixed point \( r^* = 1, \; \theta^* = 0 \) is attracting but not Liapunov stable.

6.3.16 (Saddle switching and structural stability) Consider the system \( \dot{x} = a + x^2 - xy, \; \dot{y} = y^2 - x^2 - 1 \), where \( a \) is a parameter.

a) Sketch the phase portrait for \( a = 0 \). Show that there is a trajectory connecting two saddle points. (Such a trajectory is called a saddle connection.)

b) With the aid of a computer if necessary, sketch the phase portrait for \( a < 0 \) and \( a > 0 \).

Notice that for \( a \neq 0 \), the phase portrait has a different topological character: the saddles are no longer connected by a trajectory. The point of this exercise is that the phase portrait in (a) is not structurally stable, since its topology can be changed by an arbitrarily small perturbation \( a \).

6.3.17 (Nasty fixed point) The system \( \dot{x} = xy - x^2 y + y^3, \; \dot{y} = y^2 + x^3 - xy^2 \) has a nasty higher-order fixed point at the origin. Using polar coordinates or otherwise, sketch the phase portrait.

6.4 Rabbits versus Sheep

Consider the following "rabbits vs. sheep" problems, where \( x, y \geq 0 \). Find the fixed points, investigate their stability, draw the nullclines, and sketch plausible phase portraits. Indicate the basins of attraction of any stable fixed points.

6.4.1 \( \dot{x} = x(3 - x - y), \; \dot{y} = y(2 - x - y) \)

6.4.2 \( \dot{x} = x(3 - 2x - y), \; \dot{y} = y(2 - x - y) \)

6.4.3 \( \dot{x} = x(3 - 2x - 2y), \; \dot{y} = y(2 - x - y) \)

The next three exercises deal with competition models of increasing complexity. We assume \( N_1, N_2 \geq 0 \) in all cases.

6.4.4 The simplest model is \( \dot{N}_1 = r_1 N_1 - b_1 N_1 N_2, \; \dot{N}_2 = r_2 N_2 - b_2 N_1 N_2 \).

a) In what way is this model less realistic than the one considered in the text?

b) Show that by suitable rescalings of \( N_1, N_2, \) and \( t \), the model can be nondimensionalized to \( x' = x(1 - y), \; y' = y(\rho - x) \). Find a formula for the dimensionless group \( \rho \).

c) Sketch the nullclines and vector field for the system in (b).

d) Draw the phase portrait, and comment on the biological implications.

e) Show that (almost) all trajectories are curves of the form \( \rho \ln x - x = \ln y - y + C \). (Hint: Derive a differential equation for \( dx/dy \), and separate the variables.) Which trajectories are not of the stated form?

6.4.5 Now suppose that species #1 has a finite carrying capacity \( K_1 \). Thus
\[
\dot{N}_1 = r_1 N_1 (1 - N_1 / K_1) - b_1 N_1 N_2 \\
\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2,
\]

Nondimensionalize the model and analyze it. Show that there are two qualitatively different kinds of phase portrait, depending on the size of \( K_1 \). (Hint: Draw the nullclines.) Describe the long-term behavior in each case.

**6.4.6** Finally, suppose that both species have finite carrying capacities:

\[
\dot{N}_1 = r_1 N_1 (1 - N_1 / K_1) - b_1 N_1 N_2 \\
\dot{N}_2 = r_2 N_2 (1 - N_2 / K_2) - b_2 N_1 N_2.
\]

a) Nondimensionalize the model. How many dimensionless groups are needed? b) Show that there are four qualitatively different phase portraits, as far as long-term behavior is concerned. c) Find conditions under which the two species can stably coexist. Explain the biological meaning of these conditions. (Hint: The carrying capacities reflect the competition within a species, whereas the \( b \)'s reflect the competition between species.)

**6.4.7** (Two-mode laser) According to Haken (1983, p. 129), a two-mode laser produces two different kinds of photons with numbers \( n_1 \) and \( n_2 \). By analogy with the simple laser model discussed in Section 3.3, the rate equations are

\[
\dot{n}_1 = G_1 N n_1 - k_1 n_1 \\
\dot{n}_2 = G_2 N n_2 - k_2 n_2
\]

where \( N(t) = N_0 - \alpha_1 n_1 - \alpha_2 n_2 \) is the number of excited atoms. The parameters \( G_1, G_2, k_1, k_2, \alpha_1, \alpha_2, N_0 \) are all positive. a) Discuss the stability of the fixed point \( n_1^* = n_2^* = 0 \). b) Find and classify any other fixed points that may exist. c) Depending on the values of the various parameters, how many qualitatively different phase portraits can occur? For each case, what does the model predict about the long-term behavior of the laser?

**6.5 Conservative Systems**

**6.5.1** Consider the system \( \ddot{x} = x^3 - x \).

a) Find all the equilibrium points and classify them. b) Find a conserved quantity. c) Sketch the phase portrait.

**6.5.2** Consider the system \( \ddot{x} = x - x^2 \).

a) Find and classify the equilibrium points.
b) Sketch the phase portrait.

c) Find an equation for the homoclinic orbit that separates closed and nonclosed trajectories.

6.5.3 Find a conserved quantity for the system \( \dot{x} = a - e^x \), and sketch the phase portrait for \( a < 0 \), \( a = 0 \), and \( a > 0 \).

6.5.4 Sketch the phase portrait for the system \( \dot{x} = ax - x^2 \) for \( a < 0 \), \( a = 0 \), and \( a > 0 \).

6.5.5 Investigate the stability of the equilibrium points of the system \( \dot{x} = (x - a)(x^2 - a) \) for all real values of the parameter \( a \). (Hints: It might help to graph the right-hand side. An alternative is to rewrite the equation as \( \dot{x} = -V'(x) \) for a suitable potential energy function \( V \) and then use your intuition about particles moving in potentials.)

6.5.6 (Epidemic model revisited) In Exercise 3.7.6, you analyzed the Kermack–McKendrick model of an epidemic by reducing it to a certain first-order system. In this problem you'll see how much easier the analysis becomes in the phase plane. As before, let \( x(t) \geq 0 \) denote the size of the healthy population and \( y(t) \geq 0 \) denote the size of the sick population. Then the model is

\[
\begin{align*}
\dot{x} &= -kxy, \\
\dot{y} &= kxy - \ell y
\end{align*}
\]

where \( k, \ell > 0 \). (The equation for \( z(t) \), the number of deaths, plays no role in the \( x, y \) dynamics so we omit it.)

a) Find and classify all the fixed points.

b) Sketch the nullclines and the vector field.

c) Find a conserved quantity for the system. (Hint: Form a differential equation for \( \frac{dy}{dx} \). Separate the variables and integrate both sides.)

d) Plot the phase portrait. What happens as \( t \to \infty \)?

e) Let \( (x_0, y_0) \) be the initial condition. An epidemic is said to occur if \( y(t) \) increases initially. Under what condition does an epidemic occur?

6.5.7 (General relativity and planetary orbits) The relativistic equation for the orbit of a planet around the sun is

\[
\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2
\]

where \( u = 1/r \) and \( r, \theta \) are the polar coordinates of the planet in its plane of motion. The parameter \( \alpha \) is positive and can be found explicitly from classical Newtonian mechanics; the term \( \varepsilon u^2 \) is Einstein’s correction. Here \( \varepsilon \) is a very small positive parameter.

a) Rewrite the equation as a system in the \((u, v)\) phase plane, where \( v = du/d\theta \).
b) Find all the equilibrium points of the system.
c) Show that one of the equilibria is a center in the \((u, v)\) phase plane, according to the linearization. Is it a nonlinear center?
d) Show that the equilibrium point found in (c) corresponds to a circular planetary orbit.

Hamiltonian systems are fundamental to classical mechanics; they provide an equivalent but more geometric version of Newton's laws. They are also central to celestial mechanics and plasma physics, where dissipation can sometimes be neglected on the time scales of interest. The theory of Hamiltonian systems is deep and beautiful, but perhaps too specialized and subtle for a first course on nonlinear dynamics. See Arnold (1978), Lichtenberg and Lieberman (1992), Tabor (1989), or Hénon (1983) for introductions.

Here's the simplest instance of a Hamiltonian system. Let \(H(p, q)\) be a smooth, real-valued function of two variables. The variable \(q\) is the "generalized coordinate" and \(p\) is the "conjugate momentum." (In some physical settings, \(H\) could also depend explicitly on time \(t\), but we'll ignore that possibility.) Then a system of the form

\[
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}
\]

is called a Hamiltonian system and the function \(H\) is called the Hamiltonian. The equations for \(\dot{q}\) and \(\dot{p}\) are called Hamilton's equations.

The next three exercises concern Hamiltonian systems.

**6.5.8** (Harmonic oscillator) For a simple harmonic oscillator of mass \(m\), spring constant \(k\), displacement \(x\), and momentum \(p\), the Hamiltonian is \(H = \frac{p^2}{2m} + \frac{kx^2}{2}\). Write out Hamilton's equations explicitly. Show that one equation gives the usual definition of momentum and the other is equivalent to \(F = ma\). Verify that \(H\) is the total energy.

**6.5.9** Show that for any Hamiltonian system, \(H(x, p)\) is a conserved quantity. (Hint: Show \(\dot{H} = 0\) by applying the chain rule and invoking Hamilton's equations.) Hence the trajectories lie on the contour curves \(H(x, p) = C\).

**6.5.10** (Inverse-square law) A particle moves in a plane under the influence of an inverse-square force. It is governed by the Hamiltonian \(H(p, r) = \frac{p^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r}\) where \(r > 0\) is the distance from the origin and \(p\) is the radial momentum. The parameters \(h\) and \(k\) are the angular momentum and the force constant, respectively.

a) Suppose \(k > 0\), corresponding to an attractive force like gravity. Sketch the
phase portrait in the \((r, p)\) plane. (Hint: Graph the “effective potential” \(V(r) = h^2/2r^2 - k/r\) and then look for intersections with horizontal lines of height \(E\). Use this information to sketch the contour curves \(H(p, r) = E\) for various positive and negative values of \(E\).)

b) Show that the trajectories are closed if \(-k^2/2h^2 < E < 0\), in which case the particle is “captured” by the force. What happens if \(E > 0\)? What about \(E = 0\)?

c) If \(k < 0\) (as in electric repulsion), show that there are no periodic orbits.

6.5.11 (Basins for damped double-well oscillator) Suppose we add a small amount of damping to the double-well oscillator of Example 6.5.2. The new system is \(\dot{x} = y, \dot{y} = -by + x - x^3\), where \(0 < b << 1\). Sketch the basin of attraction for the stable fixed point \((x^*, y^*) = (1, 0)\). Make the picture large enough so that the global structure of the basin is clearly indicated.

6.5.12 (Why we need to assume isolated minima in Theorem 6.5.1) Consider the system \(\dot{x} = xy, \dot{y} = -x^2\).

a) Show that \(E = x^2 + y^2\) is conserved.

b) Show that the origin is a fixed point, but not an isolated fixed point.

c) Since \(E\) has a local minimum at the origin, one might have thought that the origin has to be a center. But that would be a misuse of Theorem 6.5.1; the theorem does not apply here because the origin is not an isolated fixed point. Show that in fact the origin is not surrounded by closed orbits, and sketch the actual phase portrait.

6.5.13 (Nonlinear centers)

a) Show that the Duffing equation \(\ddot{x} + x + \varepsilon x^3 = 0\) has a nonlinear center at the origin for all \(\varepsilon > 0\).

b) If \(\varepsilon < 0\), show that all trajectories near the origin are closed. What about trajectories that are far from the origin?

6.5.14 (Glider) Consider a glider flying at speed \(v\) at an angle \(\theta\) to the horizontal. Its motion is governed approximately by the dimensionless equations

\[
\dot{v} = -\sin \theta - Dv^2 \\
\dot{\theta} = -\cos \theta + v^2
\]

where the trigonometric terms represent the effects of gravity and the \(v^2\) terms represent the effects of drag and lift.

a) Suppose there is no drag \((D = 0)\). Show that \(v^3 - 3v \cos \theta\) is a conserved quantity. Sketch the phase portrait in this case. Interpret your results physically—what does the flight path of the glider look like?

b) Investigate the case of positive drag \((D > 0)\).

In the next four exercises, we return to the problem of a bead on a rotating hoop.
discussed in Section 3.5. Recall that the bead’s motion is governed by

\[ mr\ddot{\phi} = -b\dot{\phi} - mg \sin \phi + mr\omega^2 \sin \phi \cos \phi. \]

Previously, we could only treat the overdamped limit. The next four exercises deal with the dynamics more generally.

6.5.15 (Frictionless bead) Consider the undamped case \( b = 0 \).

a) Show that the equation can be nondimensionalized to \( \ddot{\phi} = \sin \phi (\cos \phi - \gamma^{-1}) \), where \( \gamma = r\omega^2/g \) as before, and prime denotes differentiation with respect to dimensionless time \( \tau = \omega t \).

b) Draw all the qualitatively different phase portraits as \( \gamma \) varies.

c) What do the phase portraits imply about the physical motion of the bead?

6.5.16 (Small oscillations of the bead) Return to the original dimensional variables. Show that when \( b = 0 \) and \( \omega \) is sufficiently large, the system has a symmetric pair of stable equilibria. Find the approximate frequency of small oscillations about these equilibria. (Please express your answer with respect to \( t \), not \( \tau \).)

6.5.17 (A puzzling constant of motion for the bead) Find a conserved quantity when \( b = 0 \). You might think that it’s essentially the bead’s total energy, but it isn’t! Show explicitly that the bead’s kinetic plus potential energy is not conserved. Does this make sense physically? Can you find a physical interpretation for the conserved quantity? (Hint: Think about reference frames and moving constraints.)

6.5.18 (General case for the bead) Finally, allow the damping \( b \) to be arbitrary. Define an appropriate dimensionless version of \( b \), and plot all the qualitatively different phase portraits that occur as \( b \) and \( \gamma \) vary.

6.5.19 (Rabbits vs. foxes) The model \( \dot{R} = aR - bRF, \dot{F} = -cF + dRF \) is the Lotka-Volterra predator-prey model. Here \( R(t) \) is the number of rabbits, \( F(t) \) is the number of foxes, and \( a, b, c, d > 0 \) are parameters.

a) Discuss the biological meaning of each of the terms in the model. Comment on any unrealistic assumptions.

b) Show that the model can be recast in dimensionless form as \( x' = x(1 - y), \quad y' = \mu y(x - 1) \).

c) Find a conserved quantity in terms of the dimensionless variables.

d) Show that the model predicts cycles in the populations of both species, for almost all initial conditions.

This model is popular with many textbook writers because it’s simple, but some are beguiled into taking it too seriously. Mathematical biologists dismiss the Lotka-Volterra model because it is not structurally stable, and because real predator-prey cycles typically have a characteristic amplitude. In other words, realistic
models should predict a single closed orbit, or perhaps finitely many, but not a continuous family of neutrally stable cycles. See the discussions in May (1972), Edelstein-Keshet (1988), or Murray (1989).

6.6 Reversible Systems
Show that each of the following systems is reversible, and sketch the phase portrait.

6.6.1 \( \dot{x} = y(1 - x^2), \quad \dot{y} = 1 - y^2 \)

6.6.2 \( \dot{x} = y, \quad \dot{y} = x \cos y \)

6.6.3 (Wallpaper) Consider the system \( \dot{x} = \sin y, \quad \dot{y} = \sin x \).
   a) Show that the system is reversible.
   b) Find and classify all the fixed points.
   c) Show that the lines \( y = \pm x \) are invariant (any trajectory that starts on them stays on them forever).
   d) Sketch the phase portrait.

6.6.4 (Computer explorations) For each of the following reversible systems, try to sketch the phase portrait by hand. Then use a computer to check your sketch. If the computer reveals patterns you hadn't anticipated, try to explain them.
   a) \( \dot{x} + (\dot{x})^2 + x = 3 \)
   b) \( \dot{x} = y - y^3, \quad \dot{y} = x \cos y \)
   c) \( \dot{x} = \sin y, \quad \dot{y} = y^2 - x \)

6.6.5 Consider equations of the form \( \dot{x} + f(x) + g(x) = 0 \), where \( f \) is an even function, and both \( f \) and \( g \) are smooth.
   a) Show that the equation is invariant under the pure time-reversal symmetry \( t \rightarrow -t \).
   b) Show that the equilibrium points cannot be stable nodes or spirals.

6.6.6 (Manta ray) Use qualitative arguments to deduce the “manta ray” phase portrait of Example 6.6.1.
   a) Plot the nullclines \( \dot{x} = 0 \) and \( \dot{y} = 0 \).
   b) Find the sign of \( \dot{x}, \dot{y} \) in different regions of the plane.
   c) Calculate the eigenvalues and eigenvectors of the saddle points at \( (-1, \pm 1) \).
   d) Consider the unstable manifold of \( (-1, -1) \). By making an argument about the signs of \( \dot{x}, \dot{y} \), prove that this unstable manifold intersects the negative \( x \)-axis. Then use reversibility to prove the existence of a heteroclinic trajectory connecting \( (-1, -1) \) to \( (-1, 1) \).
   e) Using similar arguments, prove that another heteroclinic trajectory exists, and sketch several other trajectories to fill in the phase portrait.

6.6.7 (Oscillator with both positive and negative damping) Show that the system \( \ddot{x} + x\dot{x} + x = 0 \) is reversible and plot the phase portrait.

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6.6.8 (Reversible system on a cylinder) While studying chaotic streamlines inside a drop immersed in a steady Stokes flow, Stone et al. (1991) encountered the system

\[ \dot{x} = \frac{\sqrt{2}}{4} x (x - 1) \sin \phi, \quad \dot{\phi} = \frac{1}{2} \left[ \beta - \frac{1}{\sqrt{2}} \cos \phi - \frac{1}{8 \sqrt{2}} x \cos \phi \right] \]

where \(0 \leq x \leq 1\) and \(-\pi \leq \phi < \pi\).

Since the system is \(2\pi\)-periodic in \(\phi\), it may be considered as a vector field on a cylinder. (See Section 6.7 for another vector field on a cylinder.) The \(x\)-axis runs along the cylinder, and the \(\phi\)-axis wraps around it. Note that the cylindrical phase space is finite, with edges given by the circles \(x = 0\) and \(x = 1\).

a) Show that the system is reversible.

b) Verify that for \(\beta > \frac{1}{\sqrt{2}}\), the system has three fixed points on the cylinder, one of which is a saddle. Show that this saddle is connected to itself by a homoclinic orbit that winds around the waist of the cylinder. Using reversibility, prove that there is a band of closed orbits sandwiched between the circle \(x = 0\) and the homoclinic orbit. Sketch the phase portrait on the cylinder, and check your results by numerical integration.

c) Show that as \(\beta \to \frac{1}{\sqrt{2}}\) from above, the saddle point moves toward the circle \(x = 0\), and the homoclinic orbit tightens like a noose. Show that all the closed orbits disappear when \(\beta = \frac{1}{\sqrt{2}}\).

d) For \(0 < \beta < \frac{1}{\sqrt{2}}\), show that there are two saddle points on the edge \(x = 0\). Plot the phase portrait on the cylinder.

6.6.9 (Josephson junction array) As discussed in Exercises 4.6.4 and 4.6.5, the equations

\[ \frac{d\phi_k}{d\tau} = \Omega + a \sin \phi_k + \frac{1}{N} \sum_{j=1}^{N} \sin \phi_j, \text{ for } k = 1, 2, \ldots \]

arise as the dimensionless circuit equations for a resistively loaded array of Josephson junctions.

a) Let \(\theta_k = \phi_k - \frac{\pi}{2}\), and show that the resulting system for \(\theta_k\) is reversible.

b) Show that there are four fixed points (mod \(2\pi\)) when \(|\Omega/(a + 1)| < 1\), and none when \(|\Omega/(a + 1)| > 1\).

c) Using the computer, explore the various phase portraits that occur for \(a = 1\), as \(\Omega\) varies over the interval \(0 \leq \Omega \leq 3\).

For more about this system, see Tsang et al. (1991).

6.6.10 Is the origin a nonlinear center for the system \(\dot{x} = -y - x^2, \dot{y} = x\)?

6.6.11 (Rotational dynamics and a phase portrait on a sphere) The rotational dynamics of an object in a shear flow are governed by
\[ \dot{\theta} = \cot \phi \cos \theta, \quad \dot{\phi} = (\cos^2 \phi + A \sin^2 \phi) \sin \theta, \]

where \( \theta \) and \( \phi \) are spherical coordinates that describe the orientation of the object. Our convention here is that \(- \pi < \theta \leq \pi\) is the “longitude,” i.e., the angle around the \( z \)-axis, and \(- \frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}\) is the “latitude,” i.e., the angle measured northward from the equator. The parameter \( A \) depends on the shape of the object.

a) Show that the equations are reversible in two ways: under \( t \rightarrow -t, \theta \rightarrow -\theta \) and under \( t \rightarrow -t, \phi \rightarrow -\phi \).

b) Investigate the phase portraits when \( A \) is positive, zero, and negative. You may sketch the phase portraits as Mercator projections (treating \( \theta \) and \( \phi \) as rectangular coordinates), but it’s better to visualize the motion on the sphere, if you can.

c) Relate your results to the tumbling motion of an object in a shear flow. What happens to the orientation of the object as \( t \rightarrow \infty \)?

### 6.7 Pendulum

#### 6.7.1 (Damped pendulum) Find and classify the fixed points of \( \ddot{\theta} + b \dot{\theta} + \sin \theta = 0 \) for all \( b > 0 \), and plot the phase portraits for the qualitatively different cases.

#### 6.7.2 (Pendulum driven by constant torque) The equation \( \ddot{\theta} + \sin \theta = \gamma \) describes the dynamics of an undamped pendulum driven by a constant torque, or an undamped Josephson junction driven by a constant bias current.

a) Find all the equilibrium points and classify them as \( \gamma \) varies.

b) Sketch the nullclines and the vector field.

c) Is the system conservative? If so, find a conserved quantity. Is the system reversible?

d) Sketch the phase portrait on the plane as \( \gamma \) varies.

e) Find the approximate frequency of small oscillations about any centers in the phase portrait.

#### 6.7.3 (Nonlinear damping) Analyze \( \ddot{\theta} + (1 + a \cos \theta) \dot{\theta} + \sin \theta = 0 \), for all \( a \geq 0 \).

#### 6.7.4 (Period of the pendulum) Suppose a pendulum governed by \( \ddot{\theta} + \sin \theta = 0 \) is swinging with an amplitude \( \alpha \). Using some tricky manipulations, we are going to derive a formula for \( T(\alpha) \), the period of the pendulum.

a) Using conservation of energy, show that \( \dot{\theta}^2 = 2(\cos \theta - \cos \alpha) \) and hence that

\[
T = 4 \int_0^\alpha \frac{d\theta}{\left[2(\cos \theta - \cos \alpha)\right]^{1/2}}.
\]

b) Using the half-angle formula, show that \( T = 4 \int_0^\alpha \frac{d\theta}{\left[4(\sin^2 \frac{1}{2} \alpha - \sin^2 \frac{1}{2} \theta)\right]^{1/2}} \).

c) The formulas in parts (a) and (b) have the disadvantage that \( \alpha \) appears in both the integrand and the upper limit of integration. To remove the \( \alpha \)-dependence
from the limits of integration, we introduce a new angle $\phi$ that runs from 0 to $\frac{\alpha}{2}$ when $\theta$ runs from 0 to $\alpha$. Specifically, let $(\sin \frac{\alpha}{2} \sin \phi = \sin \frac{\alpha}{2} \theta$. Using this substitution, rewrite (b) as an integral with respect to $\phi$. Thereby derive the exact result

$$T = 4 \int_0^{\pi/2} \frac{d\phi}{\cos \frac{\alpha}{2} \theta} = 4K(\sin^2 \frac{\alpha}{2}) ,$$

where the complete elliptic integral of the first kind is defined as

$$K(m) = \int_0^{\pi/2} \frac{d\phi}{(1 - m \sin^2 \phi)^{1/2}} , \text{ for } 0 \leq m < 1 .$$

d) By expanding the elliptic integral using the binomial series and integrating term-by-term, show that

$$T(\alpha) = 2\pi \left[ 1 + \frac{1}{16} \alpha^2 + O(\alpha^4) \right] \text{ for } \alpha << 1 .$$

Note that larger swings take longer.

**6.7.5 (Numerical solution for the period)** Redo Exercise 6.7.4 using either numerical integration of the differential equation, or numerical evaluation of the elliptic integral. Specifically, compute the period $T(\alpha)$, where $\alpha$ runs from 0 to 180° in steps of 10°.

**6.8 Index Theory**

**6.8.1** Show that each of the following fixed points has an index equal to +1.

a) stable spiral  
b) unstable spiral  
c) center  
d) star  
e) degenerate node

(Unusual fixed points) For each of the following systems, locate the fixed points and calculate the index. (Hint: Draw a small closed curve $C$ around the fixed point and examine the variation of the vector field on $C$.)

**6.8.2** $\dot{x} = x^2$, $\dot{y} = y$  
**6.8.3** $\dot{x} = y - x$, $\dot{y} = x^2$

**6.8.4** $\dot{x} = y^3$, $\dot{y} = x$  
**6.8.5** $\dot{x} = xy$, $\dot{y} = x + y$

**6.8.6** A closed orbit in the phase plane encircles $S$ saddles, $N$ nodes, $F$ spirals, and $C$ centers, all of the usual type. Show that $N + F + C = 1 + S$.

**6.8.7** (Ruling out closed orbits) Use index theory to show that the system $\dot{x} = x(4 - y - x^2)$, $\dot{y} = y(x - 1)$ has no closed orbits.

**6.8.8** A smooth vector field on the phase plane is known to have exactly three closed orbits. Two of the cycles, say $C_1$ and $C_2$, lie inside the third cycle $C_3$. However, $C_1$ does not lie inside $C_2$, nor vice-versa.

a) Sketch the arrangement of the three cycles.
b) Show that there must be at least one fixed point in the region bounded by $C_1$, $C_2$, $C_3$.

6.8.9 A smooth vector field on the phase plane is known to have exactly two closed trajectories, one of which lies inside the other. The inner cycle runs clockwise, and the outer one runs counterclockwise. True or False: There must be at least one fixed point in the region between the cycles. If true, prove it. If false, provide a simple counterexample.

6.8.10 (Open-ended question for the topologically minded) Does Theorem 6.8.2 hold for surfaces other than the plane? Check its validity for various types of closed orbits on a torus, cylinder, and sphere.

6.8.11 (Complex vector fields) Let $z = x + iy$. Explore the complex vector fields $\dot{z} = z^k$ and $\dot{z} = (\bar{z})^k$, where $k > 0$ is an integer and $\bar{z} = x - iy$ is the complex conjugate of $z$.

a) Write the vector fields in both Cartesian and polar coordinates, for the cases $k = 1, 2, 3$.

b) Show that the origin is the only fixed point, and compute its index.

c) Generalize your results to arbitrary integer $k > 0$.

6.8.12 ("Matter and antimatter") There’s an intriguing analogy between bifurcations of fixed points and collisions of particles and anti-particles. Let’s explore this in the context of index theory. For example, a two-dimensional version of the saddle-node bifurcation is given by $\dot{x} = a + x^2$, $\dot{y} = -y$, where $a$ is a parameter.

a) Find and classify all the fixed points as $a$ varies from $-\infty$ to $+\infty$.

b) Show that the sum of the indices of all the fixed points is conserved as $a$ varies.

c) State and prove a generalization of this result, for systems of the form $\dot{x} = f(x, a)$, where $x \in \mathbb{R}^2$ and $a$ is a parameter.

6.8.13 (Integral formula for the index of a curve) Consider a smooth vector field $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$ on the plane, and let $C$ be a simple closed curve that does not pass through any fixed points. As usual, let $\phi = \tan^{-1}(\dot{y}/\dot{x})$ as in Figure 6.8.1.

a) Show that $d\phi = (f\, dg - g\, df)/(f^2 + g^2)$.

b) Derive the integral formula

$$I_C = \frac{1}{2\pi} \oint_C \frac{f\, dg - g\, df}{f^2 + g^2}.$$  

6.8.14 Consider the family of linear systems $\dot{x} = x\cos\alpha - y\sin\alpha$, $\dot{y} = x\sin\alpha + y\cos\alpha$, where $\alpha$ is a parameter that runs over the range $0 \leq \alpha \leq \pi$. Let $C$ be a simple closed curve that does not pass through the origin.
a) Classify the fixed point at the origin as a function of $\alpha$.
b) Using the integral derived in Exercise 6.8.13, show that $I_C$ is independent of $\alpha$.
c) Let $C$ be a circle centered at the origin. Compute $I_C$ explicitly by evaluating the integral for any convenient choice of $\alpha$. 