$$\omega = 1+\varepsilon\phi' = 1+\tfrac{3}{8}\varepsilon a^2 + O(\varepsilon^2). \qquad (57)$$

Now for the physical interpretation. The Duffing equation describes the undamped motion of a unit mass attached to a nonlinear spring with restoring force $F(x) = -x - \varepsilon x^3$. We can use our intuition about ordinary linear springs if we write $F(x) = -kx$, where the spring stiffness is now dependent on x:

$$k = k(x) = 1+\varepsilon x^2.$$

Suppose $\varepsilon > 0$. Then the spring gets *stiffer* as the displacement x increases—this is called a ***hardening spring.*** On physical grounds we'd expect it to *increase* the frequency of the oscillations, consistent with (57). For $\varepsilon < 0$ we have a ***softening spring***, exemplified by the pendulum (Exercise 7.6.15).

It also makes sense that $r' = 0$. The Duffing equation is a conservative system and for all ε sufficiently small, it has a *nonlinear center* at the origin (Exercise 6.5.13). Since all orbits close to the origin are periodic, there can be no long-term change in amplitude, consistent with $r' = 0$. ■

Validity of Two-Timing

We conclude with a few comments about the validity of the two-timing method. The rule of thumb is that the one-term approximation x_0 will be within $O(\varepsilon)$ of the true solution x for all times up to and including $t \sim O(1/\varepsilon)$, assuming that both x and x_0 start from the same initial condition. If x is a periodic solution, the situation is even better: x_0 remains within $O(\varepsilon)$ of x for *all* t.

But for precise statements and rigorous results about these matters, and for discussions of the subtleties that can occur, you should consult more advanced treatments, such as Guckenheimer and Holmes (1983) or Grimshaw (1990). Those authors use the *method of averaging*, an alternative approach that yields the same results as two-timing. See Exercise 7.6.25 for an introduction to this powerful technique.

Also, we have been very loose about the sense in which our formulas approximate the true solutions. The relevant notion is that of *asymptotic* approximation. For introductions to asymptotics, see Lin and Segel (1988) or Bender and Orszag (1978).

EXERCISES FOR CHAPTER 7

7.1 Examples

Sketch the phase portrait for each of the following systems. (As usual, r, θ denote polar coordinates.)

7.1.1 $\dot{r} = r^3 - 4r$, $\dot{\theta} = 1$ **7.1.2** $\dot{r} = r(1-r^2)(9-r^2)$, $\dot{\theta} = 1$

7.1.3 $\dot{r} = r(1-r^2)(4-r^2)$, $\dot{\theta} = 2 - r^2$ **7.1.4** $\dot{r} = r\sin r$, $\dot{\theta} = 1$

7.1.5 (From polar to Cartesian coordinates) Show that the system $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$ is equivalent to

$$\dot{x} = x - y - x(x^2 + y^2), \qquad \dot{y} = x + y - y(x^2 + y^2),$$

where $x = r\cos\theta$, $y = r\sin\theta$. (Hint: $\dot{x} = \frac{d}{dt}(r\cos\theta) = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$.)

7.1.6 (Circuit for van der Pol oscillator) Figure 1 shows the "tetrode multivibrator" circuit used in the earliest commercial radios and analyzed by van der Pol. In van der Pol's day, the active element was a vacuum tube; today it would be a semiconductor device. It acts like an ordinary resistor when I is high, but like a negative resistor (energy source) when I is low. Its current-voltage characteristic $V = f(I)$ resembles a cubic function, as discussed below.

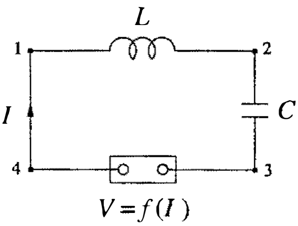

Figure 1

Suppose a source of current is attached to the circuit and then withdrawn. What equations govern the subsequent evolution of the current and the various voltages?

a) Let $V = V_{32} = -V_{23}$ denote the voltage drop from point 3 to point 2 in the circuit. Show that $\dot{V} = -I/C$ and $V = L\dot{I} + f(I)$.

b) Show that the equations in (a) are equivalent to

$$\frac{dw}{d\tau} = -x\,, \qquad \frac{dx}{d\tau} = w - \mu F(x)$$

where $x = L^{1/2} I$, $w = C^{1/2} V$, $\tau = (LC)^{-1/2} t$, and $F(x) = f(L^{-1/2} x)$.

In Section 7.5, we'll see that this system for (w, x) is equivalent to the van der Pol equation, if $F(x) = \frac{1}{3}x^3 - x$. Thus the circuit produces self-sustained oscillations.

7.1.7 (Waveform) Consider the system $\dot{r} = r(4-r^2)$, $\dot{\theta} = 1$, and let $x(t) = r(t)\cos\theta(t)$. Given the initial condition $x(0) = 0.1$, $y(0) = 0$, sketch the approximate waveform of $x(t)$, *without* obtaining an explicit expression for it.

7.1.8 (A circular limit cycle) Consider $\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$, where $a > 0$.

a) Find and classify all the fixed points.

b) Show that the system has a circular limit cycle, and find its amplitude and period.

c) Determine the stability of the limit cycle.

d) Give an argument which shows that the limit cycle is unique, i.e., there are no other periodic trajectories.

7.1.9 (Circular pursuit problem) A dog at the center of circular pond sees a duck swimming along the edge. The dog chases the duck by always swimming straight toward it. In other words, the dog's velocity vector always lies along the line connecting it to the duck. Meanwhile, the duck takes evasive action by swimming around the circumference as fast as it can, always moving counterclockwise.

a) Assuming the pond has unit radius and both animals swim at the same constant speed, derive a pair of differential equations for the path of the dog. (Hint: Use the coordinate system shown in Figure 2 and find equations for $dR/d\theta$ and $d\phi/d\theta$.) Analyze the system. Can you solve it explicitly? Does the dog ever catch the duck?

b) Now suppose the dog swims k times faster than the duck. Derive the differential equations for the dog's path.

c) If $k = \frac{1}{2}$, what does the dog end up doing in the long run?

Note: This problem has a long and intriguing history, dating back to the mid-1800s at least. It is much more difficult than similar ***pursuit problems***—there is no known solution for the path of the dog in part (a), in terms of elementary functions. See Davis (1962, pp. 113–125) for a nice analysis and a guide to the literature.

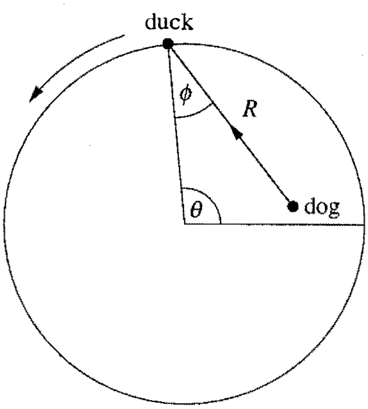

Figure 2

7.2 Ruling Out Closed Orbits

Plot the phase portraits of the following gradient systems $\dot{\mathbf{x}} = -\nabla V$.

7.2.1 $V = x^2 + y^2$ **7.2.2** $V = x^2 - y^2$ **7.2.3** $V = e^x \sin y$

7.2.4 Show that all vector fields on the line are gradient systems. Is the same true of vector fields on the circle?

7.2.5 Let $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ be a smooth vector field defined on the phase plane.

a) Show that if this is a gradient system, then $\partial f/\partial y = \partial g/\partial x$.

b) Is the condition in (a) also sufficient?

7.2.6 Given that a system is a gradient system, here's how to find its potential function V. Suppose that $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$. Then $\dot{\mathbf{x}} = -\nabla V$ implies

$f(x,y) = -\partial V/\partial x$ and $g(x,y) = -\partial V/\partial y$. These two equations may be "partially integrated" to find V. Use this procedure to find V for the following gradient systems.

a) $\dot{x} = y^2 + y\cos x, \quad \dot{y} = 2xy + \sin x$

b) $\dot{x} = 3x^2 - 1 - e^{2y}, \quad \dot{y} = -2xe^{2y}$

7.2.7 Consider the system $\dot{x} = y + 2xy$, $\dot{y} = x + x^2 - y^2$.

a) Show that $\partial f/\partial y = \partial g/\partial x$. (Then Exercise 7.2.5(a) implies this is a gradient system.)

b) Find V.

c) Sketch the phase portrait.

7.2.8 Show that the trajectories of a gradient system always cross the equipotentials at right angles (except at fixed points).

7.2.9 For each of the following systems, decide whether it is a gradient system. If so, find V and sketch the phase portrait. On a separate graph, sketch the equipotentials $V = \text{constant}$. (If the system is not a gradient system, go on to the next question.)

a) $\dot{x} = y + x^2 y, \quad \dot{y} = -x + 2xy$

b) $\dot{x} = 2x, \quad \dot{y} = 8y$

c) $\dot{x} = -2xe^{x^2+y^2}, \quad \dot{y} = -2ye^{x^2+y^2}$

7.2.10 Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits, by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.

7.2.11 Show that $V = ax^2 + 2bxy + cy^2$ is positive definite if and only if $a > 0$ and $ac - b^2 > 0$. (This is a useful criterion that allows us to test for positive definiteness when the quadratic form V includes a "cross term" $2bxy$.)

7.2.12 Show that $\dot{x} = -x + 2y^3 - 2y^4$, $\dot{y} = -x - y + xy$ has no periodic solutions. (Hint: Choose a, m, and n such that $V = x^m + ay^n$ is a Liapunov function.)

7.2.13 Recall the competition model

$$\dot{N}_1 = r_1 N_1(1 - N_1/K_1) - b_1 N_1 N_2, \qquad \dot{N}_2 = r_2 N_2(1 - N_2/K_2) - b_2 N_1 N_2,$$

of Exercise 6.4.6. Using Dulac's criterion with the weighting function $g = (N_1 N_2)^{-1}$, show that the system has no periodic orbits in the first quadrant $N_1, N_2 > 0$.

7.2.14 Consider $\dot{x} = x^2 - y - 1$, $\dot{y} = y(x - 2)$.

a) Show that there are three fixed points and classify them.

b) By considering the three straight lines through pairs of fixed points, show that there are no closed orbits.
c) Sketch the phase portrait.

7.2.15 Consider the system $\dot{x} = x(2 - x - y)$, $\dot{y} = y(4x - x^2 - 3)$. We know from Example 7.2.4 that this system has no closed orbits.
a) Find the three fixed points and classify them.
b) Sketch the phase portrait.

7.2.16 If R is not simply connected, then the conclusion of Dulac's criterion is no longer valid. Find a counterexample.

7.2.17 Assume the hypotheses of Dulac's criterion, except now suppose that R is topologically equivalent to an annulus, i.e., it has exactly one hole in it. Using Green's theorem, show that there exists *at most* one closed orbit in R. (This result can be useful sometimes as a way of proving that a closed orbit is unique.)

7.3 Poincaré-Bendixson Theorem

7.3.1 Consider $\dot{x} = x - y - x(x^2 + 5y^2)$, $\dot{y} = x + y - y(x^2 + y^2)$.
a) Classify the fixed point at the origin.
b) Rewrite the system in polar coordinates, using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.
c) Determine the circle of maximum radius, r_1, centered on the origin such that all trajectories have a radially *outward* component on it.
d) Determine the circle of minimum radius, r_2, centered on the origin such that all trajectories have a radially *inward* component on it.
e) Prove that the system has a limit cycle somewhere in the trapping region $r_1 \leq r \leq r_2$.

7.3.2 Using numerical integration, compute the limit cycle of Exercise 7.3.1 and verify that it lies in the trapping region you constructed.

7.3.3 Show that the system $\dot{x} = x - y - x^3$, $\dot{y} = x + y - y^3$ has a periodic solution.

7.3.4 Consider the system

$$\dot{x} = x(1 - 4x^2 - y^2) - \tfrac{1}{2}y(1 + x), \qquad \dot{y} = y(1 - 4x^2 - y^2) + 2x(1 + x).$$

a) Show that the origin is an unstable fixed point.
b) By considering $\dot{V}$, where $V = (1 - 4x^2 - y^2)^2$, show that all trajectories approach the ellipse $4x^2 + y^2 = 1$ as $t \to \infty$.

7.3.5 Show that the system $\dot{x} = -x - y + x(x^2 + 2y^2)$, $\dot{y} = x - y + y(x^2 + 2y^2)$ has at least one periodic solution.

7.3.6 Consider the oscillator equation $\ddot{x}+F(x,\dot{x})\dot{x}+x=0$, where $F(x,\dot{x})<0$ if $r\le a$ and $F(x,\dot{x})>0$ if $r\ge b$, where $r^2=x^2+\dot{x}^2$.

a) Give a physical interpretation of the assumptions on F.

b) Show that there is at least one closed orbit in the region $a<r<b$.

7.3.7 Consider $\dot{x}=y+ax(1-2b-r^2)$, $\dot{y}=-x+ay(1-r^2)$, where a and b are parameters ($0<a\le 1$, $0\le b<\frac{1}{2}$) and $r^2=x^2+y^2$.

a) Rewrite the system in polar coordinates.

b) Prove that there is at least one limit cycle, and that if there are several, they all have the same period $T(a,b)$.

c) Prove that for $b=0$ there is only one limit cycle.

7.3.8 Recall the system $\dot{r}=r(1-r^2)+\mu r\cos\theta$, $\dot{\theta}=1$ of Example 7.3.1. Using the computer, plot the phase portrait for various values of $\mu>0$. Is there a critical value μ_c at which the closed orbit ceases to exist? If so, estimate it. If not, prove that a closed orbit exists for *all* $\mu>0$.

7.3.9 (Series approximation for a closed orbit) In Example 7.3.1, we used the Poincaré–Bendixson Theorem to prove that the system $\dot{r}=r(1-r^2)+\mu r\cos\theta$, $\dot{\theta}=1$ has a closed orbit in the annulus $\sqrt{1-\mu}<r<\sqrt{1+\mu}$ for all $\mu<1$.

a) To approximate the shape $r(\theta)$ of the orbit for $\mu\ll 1$, assume a power series solution of the form $r(\theta)=1+\mu r_1(\theta)+O(\mu^2)$. Substitute the series into a differential equation for $dr/d\theta$. Neglect all $O(\mu^2)$ terms, and thereby derive a simple differential equation for $r_1(\theta)$. Solve this equation explicitly for $r_1(\theta)$. (The approximation technique used here is called regular perturbation theory; see Section 7.6.)

b) Find the maximum and minimum r on your approximate orbit, and hence show that it lies in the annulus $\sqrt{1-\mu}<r<\sqrt{1+\mu}$, as expected.

c) Use a computer to calculate $r(\theta)$ numerically for various small μ, and plot the results on the same graph as your analytical approximation for $r(\theta)$. How does the maximum error depend on μ?

7.3.10 Consider the two-dimensional system $\dot{\mathbf{x}}=A\mathbf{x}-r^2\mathbf{x}$, where $r=\|\mathbf{x}\|$ and A is a 2×2 constant real matrix with complex eigenvalues $\alpha\pm i\omega$. Prove that there exists at least one limit cycle for $\alpha>0$ and that there are none for $\alpha<0$.

7.3.11 (Cycle graphs) Suppose $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})$ is a smooth vector field on $\mathbf{R}^2$. An improved version of the Poincaré–Bendixson theorem states that if a trajectory is trapped in a compact region, then it must approach a fixed point, a limit cycle, or something exotic called a ***cycle graph*** (an invariant set containing a finite number of fixed points connected by a finite number of trajectories, all oriented either clockwise or counterclockwise). Cycle graphs are rare in practice; here's a contrived but simple example.

a) Plot the phase portrait for the system

$$\dot{r} = r(1-r^2)\left[r^2 \sin^2\theta + (r^2\cos^2\theta - 1)^2\right]$$
$$\dot{\theta} = r^2 \sin^2\theta + (r^2\cos^2\theta - 1)^2$$

where r, θ are polar coordinates. (Hint: Note the common factor in the two equations; examine where it vanishes.)

b) Sketch x vs. t for a trajectory starting away from the unit circle. What happens as $t \to \infty$?

7.4 Liénard Systems

† **7.4.1** Show that the equation $\ddot{x} + \mu(x^2-1)\dot{x} + \tanh x = 0$, for $\mu > 0$, has exactly one periodic solution, and classify its stability.

† **7.4.2** Consider the equation $\ddot{x} + \mu(x^4 - 1)\dot{x} + x = 0$.

a) Prove that the system has a unique stable limit cycle if $\mu > 0$.

b) Using a computer, plot the phase portrait for the case $\mu = 1$.

c) If $\mu < 0$, does the system still have a limit cycle? If so, is it stable or unstable?

7.5 Relaxation Oscillations

7.5.1 For the van der Pol oscillator with $\mu >> 1$, show that the positive branch of the cubic nullcline begins at $x_A = 2$ and ends at $x_B = 1$.

7.5.2 In Example 7.5.1, we used a tricky phase plane (often called the ***Liénard plane***) to analyze the van der Pol oscillator for $\mu >> 1$. Try to redo the analysis in the standard phase plane where $\dot{x} = y$, $\dot{y} = -x - \mu(x^2 - 1)$. What is the advantage of the Liénard plane?

7.5.3 Estimate the period of the limit cycle of $\ddot{x} + k(x^2 - 4)\dot{x} + x = 1$ for $k >> 1$.

7.5.4 (Piecewise-linear nullclines) Consider the equation $\ddot{x} + \mu f(x)\dot{x} + x = 0$, where $f(x) = -1$ for $|x| < 1$ and $f(x) = 1$ for $|x| \geq 1$.

a) Show that the system is equivalent to $\dot{x} = \mu(y - F(x))$, $\dot{y} = -x/\mu$, where $F(x)$ is the piecewise-linear function

$$F(x) = \begin{cases} x+2, & x \leq -1 \\ -x, & |x| \leq 1 \\ x-2, & x \geq 1\,. \end{cases}$$

b) Graph the nullclines.
c) Show that the system exhibits relaxation oscillations for $\mu \gg 1$, and plot the limit cycle in the (x, y) plane.
d) Estimate the period of the limit cycle for $\mu \gg 1$.

7.5.5 Consider the equation $\ddot{x} + \mu(|x| - 1)\dot{x} + x = 0$. Find the approximate period of the limit cycle for $\mu \gg 1$.

7.5.6 (Biased van der Pol) Suppose the van der Pol oscillator is biased by a constant force: $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$, where a can be positive, negative, or zero. (Assume $\mu > 0$ as usual.)
a) Find and classify all the fixed points.
b) Plot the nullclines in the Liénard plane. Show that if they intersect on the *middle* branch of the cubic nullcline, the corresponding fixed point is unstable.
c) For $\mu \gg 1$, show that the system has a stable limit cycle if and only if $|a| < a_c$, where a_c is to be determined. (Hint: Use the Liénard plane.)
d) Sketch the phase portrait for a slightly greater than a_c. Show that the system is *excitable* (it has a globally attracting fixed point, but certain disturbances can send the system on a long excursion through phase space before returning to the fixed point; compare Exercise 4.5.3.)

This system is closely related to the Fitzhugh–Nagumo model of neural activity; see Murray (1989) or Edelstein–Keshet (1988) for an introduction.

7.5.7 (Cell cycle) Tyson (1991) proposed an elegant model of the cell division cycle, based on interactions between the proteins cdc2 and cyclin. He showed that the model's mathematical essence is contained in the following set of dimensionless equations:

$$\dot{u} = b(v - u)(\alpha + u^2) - u, \qquad \dot{v} = c - u,$$

where u is proportional to the concentration of the active form of a cdc2-cyclin complex, and v is proportional to the total cyclin concentration (monomers and dimers). The parameters $b \gg 1$ and $\alpha \ll 1$ are fixed and satisfy $8\alpha b < 1$, and c is adjustable.
a) Sketch the nullclines.
b) Show that the system exhibits relaxation oscillations for $c_1 < c < c_2$, where c_1 and c_2 are to be determined approximately. (It is too hard to find c_1 and c_2 exactly, but a good approximation can be achieved if you assume $8\alpha b \ll 1$.)
c) Show that the system is excitable if c is slightly less than c_1.

7.6 Weakly Nonlinear Oscillators

7.6.1 Show that if (7.6.7) is expanded as a power series in ε, we recover (7.6.17).

7.6.2 (Calibrating regular perturbation theory) Consider the initial value problem $\ddot{x}+x+\varepsilon x=0$, with $x(0)=1$, $\dot{x}(0)=0$.

a) Obtain the exact solution to the problem.

b) Using regular perturbation theory, find x_0, x_1, and x_2 in the series expansion $x(t,\varepsilon)=x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)+O(\varepsilon^3)$.

c) Does the perturbation solution contain secular terms? Did you expect to see any? Why?

7.6.3 (More calibration) Consider the initial value problem $\ddot{x}+x=\varepsilon$, with $x(0)=1$, $\dot{x}(0)=0$.

a) Solve the problem exactly.

b) Using regular perturbation theory, find x_0, x_1, and x_2 in the series expansion $x(t,\varepsilon)=x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)+O(\varepsilon^3)$.

c) Explain why the perturbation solution does or doesn't contain secular terms.

For each of the following systems $\ddot{x}+x+\varepsilon h(x,\dot{x})=0$, with $0<\varepsilon<<1$, calculate the averaged equations (7.6.53) and analyze the long-term behavior of the system. Find the amplitude and frequency of any limit cycles for the original system. If possible, solve the averaged equations explicitly for $x(t,\varepsilon)$, given the initial conditions $x(0)=a$, $\dot{x}(0)=0$.

7.6.4 $h(x,\dot{x})=x$ **7.6.5** $h(x,\dot{x})=x\dot{x}^2$

7.6.6 $h(x,\dot{x})=x\dot{x}$ **7.6.7** $h(x,\dot{x})=(x^4-1)\dot{x}$

7.6.8 $h(x,\dot{x})=(|x|-1)\dot{x}$ **7.6.9** $h(x,\dot{x})=(x^2-1)\dot{x}^3$

7.6.10 Derive the identity $\sin\theta\cos^2\theta=\frac{1}{4}[\sin\theta+\sin 3\theta]$ as follows: Use the complex representations

$$\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2},\qquad \sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i},$$

multiply everything out, and then collect terms. This is always the most straightforward method of deriving such identities, and you don't have to remember any others.

7.6.11 (Higher harmonics) Notice the third harmonic $\sin 3(\tau+\phi)$ in Equation (7.6.39). The generation of *higher harmonics* is a characteristic feature of nonlinear systems. To find the effect of such terms, return to Example 7.6.2 and solve for x_1, assuming that the original system had initial conditions $x(0)=2$, $\dot{x}(0)=0$.

7.6.12 (Deriving the Fourier coefficients) This exercise leads you through the derivation of the formulas (7.6.51) for the Fourier coefficients. For convenience, let brackets denote the average of a function: $\langle f(\theta)\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$ for any 2π-periodic function f. Let k and m be arbitrary integers.

a) Using integration by parts, complex exponentials, trig identities, or otherwise, derive the ***orthogonality relations***

$$\langle \cos k\theta \ \sin m\theta\rangle = 0\text{, for all } k,m\,;$$

$$\langle \cos k\theta \ \cos m\theta\rangle = \langle \sin k\theta \ \sin m\theta\rangle = 0\text{, for all } k \neq m\,;$$

$$\langle \cos^2 k\theta\rangle = \langle \sin^2 k\theta\rangle = \tfrac{1}{2}\text{, for } k \neq 0\,.$$

b) To find a_k for $k \neq 0$, multiply both sides of (7.6.50) by $\cos m\theta$ and average both sides term by term over the interval $[0, 2\pi]$. Now using the orthogonality relations from part (a), show that *all the terms on the right-hand side cancel out, except the* $k = m$ *term*! Deduce that $\langle h(\theta)\cos k\theta\rangle = \frac{1}{2}a_k$, which is equivalent to the formula for a_k in (7.6.51).

c) Similarly, derive the formulas for b_k and a_0.

7.6.13 (Exact period of a conservative oscillator) Consider the Duffing oscillator $\ddot{x} + x + \varepsilon x^3 = 0$, where $0 < \varepsilon << 1$, $x(0) = a$, and $\dot{x}(0) = 0$.

a) Using conservation of energy, express the oscillation period $T(\varepsilon)$ as a certain integral.

b) Expand the integrand as a power series in ε, and integrate term by term to obtain an approximate formula $T(\varepsilon) = c_0 + c_1\varepsilon + c_2\varepsilon^2 + O(\varepsilon^3)$. Find c_0, c_1, c_2 and check that c_0, c_1 are consistent with (7.6.57).

$\boxed{\textbf{7.6.14}}$ (Computer test of two-timing) Consider the equation $\ddot{x} + \varepsilon\dot{x}^3 + x = 0$.

a) Derive the averaged equations.

b) Given the initial conditions $x(0) = a$, $\dot{x}(0) = 0$, solve the averaged equations and thereby find an approximate formula for $x(t,\varepsilon)$.

c) Solve $\ddot{x} + \varepsilon\dot{x}^3 + x = 0$ numerically for $a = 1$, $\varepsilon = 2$, $0 \le t \le 50$, and plot the result on the same graph as your answer to part (b). Notice the impressive agreement, even though ε is not small!

7.6.15 (Pendulum) Consider the pendulum equation $\ddot{x} + \sin x = 0$.

a) Using the method of Example 7.6.4, show that the frequency of small oscillations of amplitude $a << 1$ is given by $\omega \approx 1 - \frac{1}{16}a^2$. (Hint: $\sin x \approx x - \frac{1}{6}x^3$, where $\frac{1}{6}x^3$ is a "small" perturbation.)

b) Is this formula for ω consistent with the exact results obtained in Exercise 6.7.4?

7.6.16 (Amplitude of the van der Pol oscillator via Green's theorem) Here's another way to determine the radius of the nearly circular limit cycle of the van der

Pol oscillator $\ddot{x} + \varepsilon\dot{x}(x^2 - 1) + x = 0$, in the limit $\varepsilon << 1$. Assume that the limit cycle is a circle of unknown radius a about the origin, and invoke the normal form of Green's theorem (i.e., the 2-D divergence theorem):

$$\oint_C \mathbf{v}\cdot\mathbf{n}\,d\ell = \iint_A \nabla\cdot\mathbf{v}\,dA$$

where C is the cycle and A is the region enclosed. By substituting $\mathbf{v} = \dot{\mathbf{x}} = (\dot{x}, \dot{y})$ and evaluating the integrals, show that $a \approx 2$.

7.6.17 (Playing on a swing) A simple model for a child playing on a swing is

$$\ddot{x} + (1 + \varepsilon\gamma + \varepsilon\cos 2t)\sin x = 0$$

where ε and γ are parameters, and $0 < \varepsilon << 1$. The variable x measures the angle between the swing and the downward vertical. The term $1 + \varepsilon\gamma + \varepsilon\cos 2t$ models the effects of gravity and the periodic pumping of the child's legs at approximately twice the natural frequency of the swing. The question is: Starting near the fixed point $x = 0$, $\dot{x} = 0$, can the child get the swing going by pumping her legs this way, or does she need a push?

a) For small x, the equation may be replaced by $\ddot{x} + (1 + \varepsilon\gamma + \varepsilon\cos 2t)x = 0$. Show that the averaged equations (7.6.53) become

$$r' = \tfrac{1}{4} r\sin 2\phi, \qquad \phi' = \tfrac{1}{2}(\gamma + \tfrac{1}{2}\cos 2\phi),$$

where $x = r\cos\theta = r(T)\cos(t + \phi(T))$, $\dot{x} = -r\sin\theta = -r(T)\sin(t + \phi(T))$, and prime denotes differentiation with respect to slow time $T = \varepsilon t$. Hint: To average terms like $\cos 2t\,\cos\theta\,\sin\theta$ over one cycle of θ, recall that $t = \theta - \phi$ and use trig identities:

$$\begin{aligned}\langle\cos 2t\,\cos\theta\,\sin\theta\rangle &= \tfrac{1}{2}\langle\cos(2\theta - 2\phi)\,\sin 2\theta\rangle\\ &= \tfrac{1}{2}\langle(\cos 2\theta\,\cos 2\phi + \sin 2\theta\,\sin 2\phi)\,\sin 2\theta\rangle\\ &= \tfrac{1}{4}\sin 2\phi.\end{aligned}$$

b) Show that the fixed point $r = 0$ is unstable to exponentially growing oscillations, i.e., $r(T) = r_0 e^{kT}$ with $k > 0$, if $|\gamma| < \gamma_c$ where γ_c is to be determined. (Hint: For r near 0, $\phi' >> r'$ so ϕ equilibrates relatively rapidly.)

c) For $|\gamma| < \gamma_c$, write a formula for the growth rate k in terms of γ.

d) How do the solutions to the averaged equations behave if $|\gamma| > \gamma_c$?

e) Interpret the results physically.

7.6.18 (Mathieu equation and a super-slow time scale) Consider the ***Mathieu equation*** $\ddot{x} + (a + \varepsilon\cos t)x = 0$ with $a \approx 1$. Using two-timing with a slow time

$T = \varepsilon^2 t$, show that the solution becomes unbounded as $t \to \infty$ if $1 - \frac{1}{12}\varepsilon^2 + O(\varepsilon^4) \le a \le 1 + \frac{5}{12}\varepsilon^2 + O(\varepsilon^4)$.

7.6.19 (Poincaré–Lindstedt method) This exercise guides you through an improved version of perturbation theory known as the ***Poincaré–Lindstedt method***. Consider the Duffing equation $\ddot{x} + x + \varepsilon x^3 = 0$, where $0 < \varepsilon << 1$, $x(0) = a$, and $\dot{x}(0) = 0$. We know from phase plane analysis that the true solution $x(t, \varepsilon)$ is periodic; our goal is to find an approximate formula for $x(t, \varepsilon)$ that is valid for all t. The key idea is to regard the frequency ω as *unknown* in advance, and to solve for it by demanding that $x(t, \varepsilon)$ contains no secular terms.

a) Define a new time $\tau = \omega t$ such that the solution has period 2π with respect to τ. Show that the equation transforms to $\omega^2 x'' + x + \varepsilon x^3 = 0$.

b) Let $x(\tau, \varepsilon) = x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + O(\varepsilon^3)$ and $\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + O(\varepsilon^3)$. (We know already that $\omega_0 = 1$ since the solution has frequency $\omega = 1$ when $\varepsilon = 0$.) Substitute these series into the differential equation and collect powers of ε. Show that

$$O(1): x_0'' + x_0 = 0$$
$$O(\varepsilon): x_1'' + x_1 = -2\omega_1 x_0'' - x_0^3.$$

c) Show that the initial conditions become $x_0(0) = a$, $\dot{x}_0(0) = 0$; $x_k(0) = \dot{x}_k(0) = 0$ for all $k > 0$.

d) Solve the $O(1)$ equation for x_0.

e) Show that after substitution of x_0 and the use of a trigonometric identity, the $O(\varepsilon)$ equation becomes $x_1'' + x_1 = (2\omega_1 a - \frac{3}{4}a^3)\cos\tau - \frac{1}{4}a^3\cos 3\tau$. Hence, *to avoid secular terms*, we need $\omega_1 = \frac{3}{8}a^2$.

f) Solve for x_1.

Two comments: (1) This exercise shows that the Duffing oscillator has a frequency that depends on amplitude: $\omega = 1 + \frac{3}{8}\varepsilon a^2 + O(\varepsilon^2)$, in agreement with (7.6.57). (2) The Poincaré–Lindstedt method is good for approximating periodic solutions, but that's *all* it can do; if you want to explore transients or nonperiodic solutions, you can't use this method. Use two-timing or averaging theory instead.

7.6.20 Show that if we had used regular perturbation to solve Exercise 7.6.19, we would have obtained $x(t, \varepsilon) = a\cos t + \varepsilon a^3\left[-\frac{3}{8}t\sin t + \frac{1}{32}(\cos 3t - \cos t)\right] + O(\varepsilon^2)$. Why is this solution inferior?

7.6.21 Using the Poincaré–Lindstedt method, show that the frequency of the limit cycle for the van der Pol oscillator $\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0$ is given by

$\omega = 1 - \frac{1}{16}\varepsilon^2 + O(\varepsilon^3)$.

7.6.22 (Asymmetric spring) Use the Poincaré–Lindstedt method to find the first few terms in the expansion for the solution of $\ddot{x} + x + \varepsilon x^2 = 0$, with $x(0) = a$, $\dot{x}(0) = 0$. Show that the center of oscillation is at $x \approx \frac{1}{2}\varepsilon a^2$, approximately.

7.6.23 Find the approximate relation between amplitude and frequency for the periodic solutions of $\ddot{x} - \varepsilon x\dot{x} + x = 0$.

7.6.24 (Computer algebra) Using Mathematica, Maple, or some other computer algebra package, apply the Poincaré–Lindstedt method to the problem $\ddot{x} + x - \varepsilon x^3 = 0$, with $x(0) = a$, and $\dot{x}(0) = 0$. Find the frequency ω of periodic solutions, up to and including the $O(\varepsilon^3)$ term.

7.6.25 (The method of averaging) Consider the weakly nonlinear oscillator $\ddot{x} + x + \varepsilon h(x, \dot{x}, t) = 0$. Let $x(t) = r(t)\cos(t + \phi(t))$, $\dot{x} = -r(t)\sin(t + \phi(t))$. This change of variables should be regarded as a definition of $r(t)$ and $\phi(t)$.

a) Show that $\dot{r} = \varepsilon h \sin(t + \phi)$, $r\dot{\phi} = \varepsilon h \cos(t + \phi)$. (Hence r and ϕ are slowly varying for $0 < \varepsilon << 1$, and thus $x(t)$ is a sinusoidal oscillation modulated by a slowly drifting amplitude and phase.)

b) Let $\langle r \rangle(t) = \bar{r}(t) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} r(\tau)\,d\tau$ denote the running average of r over one cycle of the sinusoidal oscillation. Show that $d\langle r \rangle/dt = \langle dr/dt \rangle$, i.e., it doesn't matter whether we differentiate or time-average first.

c) Show that $d\langle r \rangle/dt = \varepsilon\langle\, h[r\cos(t + \phi),\ -r\sin(t + \phi),\ t]\,\sin(t + \phi)\,\rangle$.

d) The result of part (c) is exact, but not helpful because the left-hand side involves $\langle r \rangle$ whereas the right-hand side involves r. Now comes the key approximation: replace r and ϕ by their averages over one cycle. Show that $r(t) = \bar{r}(t) + O(\varepsilon)$ and $\phi(t) = \bar{\phi}(t) + O(\varepsilon)$, and therefore

$$d\bar{r}/dt = \varepsilon\left\langle\, h[\bar{r}\cos(t + \bar{\phi}),\ -\bar{r}\sin(t + \bar{\phi}),\ t]\sin(t + \bar{\phi})\,\right\rangle + O(\varepsilon^2)$$

$$\bar{r}\,d\bar{\phi}/dt = \varepsilon\left\langle\, h[\bar{r}\cos(t + \bar{\phi}),\ -\bar{r}\sin(t + \bar{\phi}),\ t]\cos(t + \bar{\phi})\,\right\rangle + O(\varepsilon^2)$$

where the barred quantities are to be treated as constants inside the averages. These equations are just the *averaged equations* (7.6.53), derived by a different approach in the text. It is customary to drop the overbars; one usually doesn't distinguish between slowly varying quantities and their averages.

7.6.26 (Calibrating the method of averaging) Consider the equation $\dot{x} = -\varepsilon x \sin^2 t$, with $0 \le \varepsilon << 1$ and $x = x_0$ at $t = 0$.

a) Find the *exact* solution to the equation.

b) Let $\bar{x}(t) = \frac{1}{2\pi}\int_{t-\pi}^{t+\pi} x(\tau)d\tau$. Show that $x(t) = \bar{x}(t) + O(\varepsilon)$. Use the method of averaging to find an approximate differential equation satisfied by $\bar{x}$, and solve it.

c) Compare the results of parts (a) and (b); how large is the error incurred by averaging?