continuous QR for computing Lyap. Exp. (from [Geist:90])

System: $\dot{y} = v(y)$, linearization: $\dot{Y} = J|_y Y$ ① $\quad Y(0) = \mathbb{1} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \\ \vdots & 0 & \\ & \vdots & \end{bmatrix}$

QR decomp: $Y = QR$ ② $\quad Q$ orthonormal, $R =$ up. triag. $\quad (Q^TQ = 1)$ QQ^T part x part y

② in ① $\Rightarrow$ $\dot{(QR)} = JQR = \dot{Q}R + Q\dot{R} = JQR$ ③

Def: $S = Q^T\dot{Q} \Rightarrow \boxed{\dot{Q} = QS}$

lower triag of S: Q^T ③ $R^{-1} \Rightarrow Q^T\dot{Q} + \dot{R}R^{-1} = Q^TJQ$

$$\therefore\ S = Q^T\dot{Q} = Q^TJQ - \dot{R}R^{-1} \quad ④$$

But since R is up $\Delta \Rightarrow \dot{R}$ is up Δ and R^{-1} is up $\Delta \Rightarrow \dot{R}R^{-1}$ is up Δ

$\therefore$ ④ $\Rightarrow S = \left[Q^TJQ\right] - \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right] \Rightarrow \boxed{\text{lower } \Delta \text{ of } S = Q^TJQ}$ Ⓐ

upr triag of S: $Y = QR \Rightarrow Q^TY = R \Rightarrow \dot{Q}^TY + Q^T\dot{Y} = \dot{R}$

$\Rightarrow \dot{Q}^TQR + Q^TJY = \dot{R} \Rightarrow \dot{Q}^TQR + Q^TJQR = \dot{R}$

$\Rightarrow S^TRR^{-1} + Q^TJQRR^{-1} = \dot{R}R^{-1} \Rightarrow S^T = -Q^TJQ + \dot{R}R^{-1}$ Ⓑ

$\Rightarrow S = -(Q^TJQ)^T + (\dot{R}R^{-1})^T$

But $(\dot{R}R^{-1})^T$ is low Δ $\Rightarrow S = \left[-(Q^TJQ)^T\right] + \left[\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right] \Rightarrow \boxed{\text{upr } \Delta \text{ of } S = -(Q^TJQ)^T}$

diag of S: from Ⓐ & Ⓑ $\Rightarrow \mathrm{diag}(S) = \mathrm{diag}(Q^TJQ - \dot{R}R^{-1}) = \underbrace{\mathrm{diag}(-(Q^TJQ)^T + (\dot{R}R^{-1})^T)}_{ⓑ}$

but $\mathrm{diag}(Q^TJQ - \dot{R}R^{-1}) = \underbrace{\mathrm{diag}((Q^TJQ)^T - (\dot{R}R^{-1})^T)}_{ⓐ}$ $\quad \therefore$ ⓐ = ⓑ $\Rightarrow \mathrm{diag} = 0$

Therefore: $\boxed{S = \begin{bmatrix} 0 & -(Q^TJQ)^T \\ Q^TJQ & 0 \end{bmatrix}}$ (22)

Diag of R: ④ $\Rightarrow \mathrm{diag}(S) = \mathrm{diag}(Q^TJQ - \dot{R}R^{-1}) = 0 \Rightarrow \boxed{\mathrm{diag}(\dot{R}R^{-1}) = \mathrm{diag}(Q^TJQ)}$

But R is up $\Delta \Rightarrow (R^{-1})_{ii} = \dfrac{1}{R_{ii}} \overset{(24)}{\Longrightarrow} \dfrac{\dot{R}_{ii}}{R_{ii}} = (Q^TJQ)_{ii}$

Lyap exp: $\lambda_i = \lim_{t\to\infty} \dfrac{\rho_i(t)}{t}$ where $\rho_i = \ln(R_{ii}) \Rightarrow \boxed{\dot{\rho}_i = \dfrac{\dot{R}_{ii}}{R_{ii}} = (Q^TJQ)_{ii}}$

To compute Lyap. Exp.:

integrate
$$\boxed{\begin{array}{ll} N\cdot & \dot{y} = v(y) \checkmark \\ N^2\cdot & \dot{Q} = QS \checkmark \\ N\cdot & \dot{\rho}_i = (Q^TJQ)_{ii} \end{array}}$$
system with S def in (22)

I.C.s: • $Y(0) = \mathbb{1} \Rightarrow Q(0)R(0) = \mathbb{1} \Rightarrow \begin{cases} \boxed{Q(0) = \mathbb{1}} \\ R(0) = \mathbb{1} \end{cases}$

• $S(0) = (22)$ @ $t = 0$

• $\rho_{ii} =$ anything (take $= 1$)

Then $\lambda_i = \lim_{t\to\infty} \dfrac{\rho_i(t)}{t}$