HW#1 — Math 639 — Nonlinear Waves — Fall 2022

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1. Consider the *bidirectional* linear wave equation

$$u_{tt} - c^2 u_{xx} = 0. \tag{1}$$

with $c \geq 0$.

a) Find the solution to (1) if $u(x,0) = \sin(x)/x$ and $u_t(x,0) = 0$.

b) Plot $u(x,t)$ in a) for $c = 1$ on $(x,t) \in [-25:25] \times [-10:10]$.

c) Find the solution to (1) if $u(x,0) = \exp(-x^2)$ and $u_t(x,0) = 1/(x^2+1)$.

d) Plot $u(x,t)$ in c) for $c = 1$ on $(x,t) \in [-15:15] \times [-10:10]$.

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2. This exercise aims to analyze the propagation of plane waves for the linear wave equation with dissipation *and* dispersion:

$$u_t + cu_x - \gamma u_{xx} + \delta u_{xxx} = 0, \tag{2}$$

with $c, \gamma, \delta \geq 0$.

a) Consider the plane wave $u(x,t) = A\exp[i(kx - \omega t)]$. Derive the dispersion relation for this plane wave.

b) Explain in detail what happens to plane wave solutions for the different choices of k, c, γ, δ.

c) Plot, using Matlab, 4 different exact solutions for different choices of k, reflecting qualitatively different evolutions (i.e. $k = 0$, positive velocity, zero velocity and negative velocity), for $A = 1$, $c = 8$, $\delta = 2$, $\gamma = 0.5$, $x \in [-10, 10]$ and $t \in [0, 2]$. Also plot a nontrivial linear combination of these 4 solutions (cf. `dispersion.m`).

d) Rescale t, x, u in order to reduce the number of parameters (c,γ,δ) in Eq. (2). How many parameters the reduced system has? What are the implications?

e) Repeat d) for the *nonlinear* dispersive, dissipative wave equation:

$$u_t + cu_x + \beta u u_x - \gamma u_{xx} + \delta u_{xxx} = 0. \tag{3}$$

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3. Given the wave equation with dissipation *and* dispersion:

$$u_t + cu_x - \gamma u_{xx} + \delta u_{xxx} = 0, \tag{4}$$

with $c, \gamma, \delta \geq 0$.

a) Write an explicit solution for $u(x,t)$ if

$$u(x,0) = 3\cos^2(x) + \sin(x).$$

[Hint: is $\cos^2$ a plane wave? Can you make it one?]

b) Plot, using Matlab (see for example `dissipation.m`), the exact solution of a) with $c = 4$, $\delta = 1$, $\gamma = 0.25$, $x \in [0, 6\pi]$ and $t \in [0, 6]$.

c) Also plot this solution by integrating the wave equation using `wave_integrator.m`, and corroborate that you obtain the same behavior as in b).

d) The above solution seems to tend to a *single* damped plane wave displaced from zero. Explain why. Find this damped plane wave and plot it, using Matlab (see for example `dissipation.m`), for the same parameters as in b). **Caution:** When using the code `dissipation.m` you need to be aware that it is a spectral code and thus the solutions will be periodic. Therefore your initial condition must be periodic also and fit an integer number of periods in the domain. In the present case $u(x, 0)$ is indeed periodic of period 2π and this period does divide the total length of the domain (6π).